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Discontinuous Boundary Value Problems of the First Kind for Poisson's Equation.*

By Griffith C. Evans.

1. *Introduction.* Let $\Phi(e)$ be a completely additive function of point sets, representing thus a general mass distribution, defined in the open plane region T which, for simplicity, we take as bounded and simply connected. Let $\Phi(w)$ be the corresponding additive and bounded function of regular curves w in T, with regular discontinuities. We have

$$\Phi(w) = \int_T q(w, P)\, d\Phi(e_P), \tag{1}$$

where $q(w, P)$ is the density function of the region ω, bounded by w, with respect to P:

$$\begin{aligned} q(w, P) &= 1 \quad , \quad P \text{ inside } w, \\ &= 0 \quad , \quad P \text{ outside } w, \\ &= \psi/2\pi \, , \quad P \text{ on } w, \end{aligned}$$

ψ being the angle from the forward to the backward tangent at P.†

The equation to be considered is what we shall call Poisson's equation:

$$\int_w D_n u \, ds = \Phi(w), \tag{2}$$

although what is commonly known as Poisson's equation is the special case of this where $\Phi(w)$ is absolutely continuous. In fact, if $\Phi(e)$ or $\Phi(w)$ is the integral of some point function $\phi(M)$ and $\phi(M)$ is suitably restricted, so as to imply the existence and integrability of the second derivatives, the equation (2) reduces to the usual form

$$\nabla^2 u = -\phi(M). \tag{2'}$$

But we assume no specialization of this sort, since we allow $\Phi(e)$ to represent,

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* Preliminary report presented to the American Mathematical Society, September, 1927.

† H. E. Bray and G. C. Evans, "A Class of Functions Harmonic in the Sphere," *American Journal of Mathematics*, Vol. 49 (1927), pp. 153-180. See p. 169. For brevity in reference this memoir will be cited as I.

as we have said, the most general possible distribution of finite positive and negative mass on T.

We shall obtain solutions of the first boundary value problem and its generalization, in the sense of our earlier memoirs on Laplace's equation.* Somewhat analogous results may be obtained for three dimensions, but for a restricted class of regions. It may be remarked that the problems proposed for (2) are in general simpler to treat than the traditional ones for (2′), at the same time that they are wider in scope. In this sense it may be said that the situation is "correctly stated."

We need assume that (2) is required to hold merely on some special class of curves, e. g., on *almost all two-dimensional segments* s in T, that is to say, on all the rectangles formed from lines $x = a$, $y = b$ except possibly those which correspond to values of a or b respectively forming sets of linear measure zero. It will then follow that (2) will hold on more extended classes of curves. We shall be concerned especially with curves w of class Γ,† that is, simple regular curves such that for each curve there is a constant Γ with the property

$$\int_w \frac{|\cos n, MP|}{MP}\, ds_P < \Gamma,$$

holding irrespective of the position of M, n being the interior normal to the curve at P. And we shall say that a property holds for *almost all* curves of class Γ if it holds for all curves of the class except those which, from some given set E of superficial measure zero, contain portions, on their arcs, which are of positive exterior measure. The two definitions of "almost all" just given, for segments and curves respectively, are consistent.

The quantity $D_n u$ is also used in the sense of our previous memoirs. It is not always the derivative of u in the direction n, as usually defined, although for a fixed direction n it is identical with that derivative, if the latter is summable superficially, except on a set of measure zero. For the purposes of this paper, if the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ exist almost everywhere and are summable superficially, $D_\alpha u$ may be defined as the vector

$$D_\alpha u = (\partial u/\partial x) \cos x, \alpha + (\partial u/\partial y) \cos y, \alpha. \tag{3}$$

A more symmetrical definition is the following:

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* G. C. Evans, *Logarithmic Potential—Generalized Dirichlet and Neumann Problems*, New York (1927); see Ch. V. This book will be cited as II.

† G. C. Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329, cited hereafter as III. See p. 261.

(3′) $$D_\alpha u = \lim_{w \to 0} (1/\omega) \int_w u \, d\alpha',$$

where α' is the direction $\pi/2$ in advance of α and ω is the region interior to w, approaching zero as a regular family (see III, p. 274). For the functions used in this paper the two definitions would agree except possibly on a set of superficial measure zero.

2. *The Principal Solution.* We take M, a point in T, and consider what we call the *principal solution*

(4) $$U(M) = (1/2\pi) \int_T g(M,P) \, d\Phi(e_P),$$

where $g(M,P)$ is the Green's function for T with pole at P, and differs from $\log 1/MP$ by a function which is harmonic, and, for a fixed M, bounded in T. Moreover, the directional derivative or gradient of $g(M,P)$ as a function of P satisfies a relation of the form

$$\int_{\sigma_1} | \nabla g(M,P) |^2 \, d\sigma = \int_{\sigma_1} \{(\partial g/\partial x)^2 + (\partial g/\partial y)^2\} \, d\sigma$$
$$= \int_{\sigma_1} \frac{d(g,h)}{d(x,y)} \, d\sigma \leq 2\pi g_1$$

for any region σ_1 in T which does not contain a neighborhood of M, g_1 being the largest value of $g(M,P)$ for P in σ_1, and $h = h(M,P)$ being the function conjugate to $g(M,P)$. It follows then that we have the inequality:

(4′) $$\int_T | \nabla g(M,P) | \, d\sigma < A,$$

where A is some finite number depending on T.

For the circle, the Green's function may of course be expressed directly. If we let Σ be the region interior to the unit circumference C with center 0, M a point of Σ and M' its inverse with respect to C, we may write

(5) $$\begin{aligned} G(M,P) &= \log (M'P \cdot OM/MP), && M, P \text{ in } \Sigma, \ M \text{ not at } 0, \\ &= 0, && M \text{ or } P \text{ on } C, \\ &= \log (1/OP), && M \text{ at } 0. \end{aligned}$$

This is a function whose symmetry with respect to M and P as well as whose identity with the Green's function follow at once from the similarity of triangles.

The integral (4) is to be interpreted as a generalized Stieltjes or Daniell integral, since the part

$$L(M) = \int_T \log(1/MP)\, d\Phi(e_P)$$

is so given (see III, p. 257). The quantities U, $\partial U/\partial x$, $\partial U/\partial y$, $D_a U$ are defined almost everywhere in T, and although they need not remain bounded or even finite are summable over any region contained in T, and over T itself. The proof of these facts is, with the help of (4′), the same as for the case of $L(M)$ which is given in the memoir just cited. Finally, we have

$$\int_\omega D_a U(M)\, d\omega_M = \int_w U(M)\, d\alpha_M', \tag{6}$$

where α' is defined as before, valid for all rectangles, and in fact for all curves of class Γ for which $\int_w \log MP\, ds_P$ is defined when M is on w.*

If we let E_0 be the set of points, of superficial measure zero, where the relation

$$D_a U = \int_T \frac{\partial g(M,P)}{\partial \alpha_M}\, d\Phi(e_P), \quad \alpha \text{ arbitrary},$$

may fail to have meaning, the identity

$$\int_w D_n U(M)\, ds_M = \int_T d\Phi(e_P) \int_w \frac{\partial g(M,P)}{\partial n_M}\, ds_M$$
$$= \int_T q(w,P)\, d\Phi(e_P)$$

or

$$\int_w D_n U(M)\, ds_M = \Phi(w) \tag{7}$$

can fail to be valid only on those segments or on those curves of class Γ which are made up in part of subsets of E_0 of positive exterior linear measure (see III, p. 285, with correction II, p. 146).

Equation (6) states that $U(M)$ is a *potential function for its vector derivative* or gradient vector $D_a U$ in the sense of our previous memoirs,† and (7) that it satisfies (2). We have then the theorem:

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* Or in fact for all curves of class Γ if a principal value is taken, when necessary, for the improper integral over w; see III, p. 264.

† III, p. 274. See also Evans, "Note on a Theorem of Bôcher," *American Journal of Mathematics*, Vol. 50 (1928), p. 123-126 at p. 124. This note will be cited as IV.

THEOREM 1. *The principal solution* (4) *is a potential function for its vector derivative* $D_\alpha U$ *and satisfies the Poisson equation* (2) *for almost all curves of class* Γ *in* T*; in particular for almost all rectangles and almost all circles.*

2. 1. If $\Phi(e)$ is a function of positive type, corresponding therefore to an arbitrary distribution of finite positive mass, the $U(M)$ given by (4) will obviously be $\geqq 0$ or positively infinite. A characteristic property of the principal solution is described in the following theorem:

THEOREM 2. *Let* M_0 *be a point of* T *and* C_ρ *a circumference of center* M_0 *and radius* ρ *small enough to lie entirely in* T*; let* $M_1 = (\rho_1, \theta_1)$ *be a generic point of* C_ρ *when* $\rho = \rho_1$. *Then, if* $\Phi(e)$ *is of positive type,*

$$(8) \qquad U(M_0) = (1/2\pi) \int_0^{2\pi} U(M_1)\, d\theta_1 + \lim_{\rho_2 = 0} (1/2\pi) \int_{\rho_2}^{\rho_1} (1/\rho)\Phi(C_\rho)\, d\rho,$$

admitting $+\infty$ *as a possible value of* $U(M_0)$.

Hence $U(M)$ *cannot have a strict minimum at any point of* T.

It is only the first part of the theorem which requires proof. If we form the quantity $D_\rho U(M)$, we have, for almost all ρ,

$$\rho \int_{C_\rho} D_\rho U(M)\, d\theta = -\Phi(C_\rho)$$

by Theorem 1. But if we form the integral of $D_\rho U(M)$ over the annular region $\rho_2 < \rho < \rho_1$, we have for it the value (III, p. 282 and p. 288):

$$\int_{\rho_2}^{\rho_1} d\rho \int_{C_\rho} D_\rho U(M)\, d\theta = \int_0^{2\pi} d\theta \int_{\rho_2}^{\rho_1} D_\rho U(M)\, d\rho$$

$$= \int_{C_{\rho_1}} U(M_1)\, d\theta - \int_{C_{\rho_2}} U(M_2)\, d\theta,$$

and therefore

$$(9) \qquad \int_{C_{\rho_2}} U(M_2)\, d\theta = \int_{C_{\rho_1}} U(M_1)\, d\theta + \int_{\rho_2}^{\rho_1} (1/\rho)\Phi(C_\rho)\, d\rho$$

It remains merely to show that the limit of the left-hand member of (9) as ρ_2 approaches zero is $2\pi U(M_0)$.

We form

$$\int_{C_\rho} U(M)\, d\theta = (1/2\pi) \int_T d\Phi(e_P) \int_{C_\rho} g(M, P)\, d\theta_M,$$

the change in the order of integration being justifiable (see the similar integrals, III, p. 264). But

$$g(M, P) = \log (1/MP) + a(M, P)$$

where $a(M, P)$ is harmonic for M, P in T. We have

$$\int_0^{2\pi} a(M, P)\, d\theta_M = 2\pi\, a(M_0, P)$$

$$\int_0^{2\pi} \log (1/MP)\, d\theta_M = 2\pi\, k_\rho(M_0, P),$$

where

$$\begin{aligned} k_\rho(M_0, P) &= -\log M_0P, && M_0P \geq \rho \\ &= -\log \rho, && M_0P < \rho \end{aligned}$$

and yields thus an increasing sequence of continuous functions whose limit as ρ tends to zero is $-\log M_0P$. Hence

$$\int_{C\rho} U(M)\, d\theta = \int_T k_\rho(M_0, P)\, d\Phi(e_P) + \int_T a(M_0, P)\, d\Phi(e_P)$$

and

$$\begin{aligned} \lim_{\rho=0} \int_{C\rho} U(M)\, d\theta &= \int_T \log (1/M_0P)\, d\Phi(e_P) + \int_T a(M_0, P)\, d\Phi(e_P) \\ &= \int_T g(M_0, P)\, d\Phi(e_P), \end{aligned}$$

by definition of the Daniell or generalized Stieltjes integral.

But the right-hand member is precisely $2\pi U(M_0)$, and the point is proved.

It will be noticed that equation (9) is valid whether or not $\Phi(e)$ is of positive type, and if $\Phi(e)$ is written, as it may be, as the difference of two functions of positive type the rest of the analysis may be carried through provided the integral in (4) is convergent. *Equation* (8) *is valid for a general distribution of mass* $\Phi(e)$ *for points* M_0 *where* (4) *is a convergent integral, and that is, almost everywhere in* T.

2.2. Another characteristic property of the principal solution (4), when $\Phi(e)$ is of positive type, is that it is *lower semi-continuous.*

THEOREM 3. *If* $\Phi(e)$ *is of positive type, and* $U(M)$, *given by* (4), *is* $> N$ *at some point* M_0 *of* T, *then there is a neighborhood about* M_0 *for which* $U(M) > N$, M *in this neighborhood.*

We admit the possible value $U(M_0) = +\infty$, in which case N would be any positive number. The theorem requires proof only if $\Phi(T) > 0$, since if $\Phi(T) = 0$, $U(M)$ vanishes identically.

We consider the function

$$V(M) = (1/2\pi) \int_T \log (1/MP) d\Phi(e_P),$$

since $U(M) - V(M)$ is continuous in T; if the property holds for $V(M)$ it will also hold for $U(M)$. Let us assume then that $V(M_0) > N'$ and show that $V(M) > N'$ for a sufficienly small neighborhood; take $\epsilon < V(M_0) - N'$.

With the same notation as for Theorem 3, we take ρ small enough so that, whether or not the integral $V(M_0)$ converges to a finite value, we have

$$\int_T k_\rho(M_0, P) d\Phi(e_P) > N' + (\epsilon/2).$$

We take then M within $C\rho_1$, $\rho_1 < \rho$, where ρ_1 is small enough to insure the validity of the inequality

$$\log (1/MP) > k_\rho(M_0, P) - \epsilon/2\Phi(T).$$

This inequality will in fact be satisfied if ρ_1 is small enough so that

$$(MP/M_0P) < \epsilon^\eta, \quad \eta = \epsilon/2\Phi(T),$$

when M is on C_{ρ_1} and P is on $C\rho$.

It follows then that

$$\int_T \{ \log (1/MP) - k_\rho(M_0, P) \} \, d\Phi(e_P) > -(\epsilon/2\Phi(T)) \int_T d\Phi(e),$$

or

$$V(M) > \int_T k_\rho(M_0, P) \, d\Phi(e_P) - (\epsilon/2).$$

Hence, by means of the earlier inequality on the integral of the second member,

$$V(M) > N',$$

and the point is proved.

2.3. We may draw an immediate corollary from Theorems 2 and 3 with respect to solutions of Poisson's equation (2), in general. The difference of any two functions in T which are potential functions of their vector derivatives and satisfy (2) for almost all segments (or almost all circles, or almost

all curves of class Γ) will be such a solution of (2), with $\Phi(w) \equiv 0$. But a theorem which the author has already established (IV, p. 123) states that such functions are, except for removable discontinuities on a set of at most superficial measure zero, continuous with all their derivatives and solutions of Laplace's equation. We may state then the following proposition:

COROLLARY. *Let $u(M)$ be a potential function for its vector derivative, and a solution of* (2), *with $\Phi(e) \geqq 0$, for almost all segments* (*or almost all circles, or almost all curves of class* Γ) *in T. By changing the value of $u(M)$ at most in the points of a set of superficial measure zero it may be made lower semi-continuous and to satisfy* (8); *it will not then have a strict minimum at any point of T.*

3. *The class and boundary values of $U(M)$ for the circle.* As far as boundary values are concerned, it is perhaps easier to handle the problem first for the unit circle Σ, and obtain the results for other simply connected regions by conformal transformation. In this section, we show that $U(M)$ for Σ is of the class (ii), and therefore of the class (i) of the author's treatment of Laplace's equation (see II, Ch. III), and that it takes on zero boundary values for almost all points on the circumference C as M approaches C along a radius.

Let M have polar coordinates r, θ with respect to the center O of the circle, and form the integral

$$\int_{\theta'}^{\theta''} U(r,\theta)\,d\theta = (1/2\pi)\int_{\theta'}^{\theta''} d\theta \int_{\Sigma} g(r,\theta;P)\,d\Phi(e_P)$$
$$= (1/2\pi)\int_{\Sigma} d\Phi(e_P)\int_{\theta'}^{\theta''} g(r,\theta;P)\,d\theta,$$

writing $g(r,\theta;P)$ now for the Green's function (5) for C. The quantity

$$\int_{\theta'}^{\theta''} g(r,\theta;P)\,d\theta$$

is a continuous function of P, bounded independently of r, and the limit of it is zero as r tends to 1, for any P in Σ. Hence by a fundamental property of the generalized integral:

$$\lim_{r=1}\int_{\theta'}^{\theta''} U(r,\theta)\,d\theta = 0. \tag{10}$$

We need however a still stronger result.

Consider a finite number n of non-overlapping intervals (θ_i', θ_i'') at values of r, $r = r_i$, all $> r_0$ where r_0 is some value of r which will eventually be allowed to approach 1, and write

$$2\pi I = \sum_1^n{}_i\, 2\pi \int_{\theta_i'}^{\theta_i''} U(r_i, \theta)\, d\theta = \sum_1^n{}_i \int_\Sigma d\Phi(e_P) \int_{\theta_i'}^{\theta_i''} g(r_i, \theta; P)\, d\theta.$$

We shall prove that

(11) $$\lim_{r_0=1} I = 0.$$

Again, but temporarily, take $\Phi(e)$ as of positive type so that $U(r, \theta) \geq 0$. Let $r_0 = 1 - x$ and draw a second circumference C_ρ with center O and smaller radius, $\rho = 1 - (x)^{1/2}$; denote by Σ_ρ the open circular region bounded by C_ρ. Consider the integrals I_ρ and $I_{1-\rho}$ extended respectively over Σ_ρ and $\Sigma - \Sigma_\rho$, writing

$$I = I_\rho + I_{1-\rho}.$$

In Σ_ρ we have, by (5),

$$g(r, \theta; P) = g(M, P) = \log\, (OM \cdot M'P)/MP$$
$$\leq \log \frac{x^{1/2} + x - x^{3/2}}{x^{1/2} - x}.$$

for the upper bound of $g(M, P)$ when $OP < \rho$ and $OM > r_0$ will be the value of $g(M, P)$ when P, M, M' are collinear and P is on C_ρ while M is on C_{r_0}. But this quantity tends to 0 with x. Hence, given ϵ, we can take r_0 near enough to 1 so that

$$g(M, P) < 2\pi\epsilon \quad \begin{cases} r > r_0 \\ OP < \rho \end{cases}$$

Accordingly

$$I_\rho < \epsilon\, \Phi(\Sigma_\rho) \leq \epsilon\, \Phi(\Sigma)$$

and approaches zero as r_0 approaches 1.

On the other hand, if P lies in $\Sigma - \Sigma_\rho$, the integral

$$J = \Sigma \int_{\theta_i'}^{\theta_i''} g(M, P)\, d\theta_M$$

is bounded, independently of r_0, P as r_0 approaches 1. For

$$g(M, P) \leq \log 2 - \log MP \leq \log 2 - \log M_1P,$$

where M_1P is the numerical length of the perpendicular to OM from P. In other words, M_1 is a projection of M along OM on the circumference of diameter OP.

Hence

$$J \leq 2\pi \log 2 + \int_0^{2\pi} | \log M_1P \mid d\theta_M,$$

which quantity will be seen to be bounded. In fact

$$M_1P = OP \mid \sin OM, OP \mid \; = \; OP \mid \sin \psi \mid$$

where ψ is the angle OP, OM and $d\psi = d\theta$, and accordingly

$$\int_0^{2\pi} | \log M_1P \mid d\theta_M \leq 2\pi \mid \log OP \mid - 2 \int_c^{\pi} \log \sin \psi \, d\psi$$

$$\leq 2\pi \mid \log OP \mid - 4 \int_0^{\pi/2} \log \sin \psi \, d\psi$$

$$\leq 2\pi \mid \log OP \mid - 4 \int_0^{\pi/2} \log (\psi/2) \, d\psi$$

which is bounded as r_0 approaches 1 for P in $\Sigma - \Sigma_\rho$.

Hence there is a constant K, independent of r_0 as r_0 tends to 1 such that

$$I_{1-\rho} \leq K \int_{\Sigma-\Sigma_\rho} d\Phi(e_P)$$

$$\leq K \, (\Phi(\Sigma) - \Phi(\Sigma_\rho)).$$

But this also approaches zero as r_0 approaches 1, since Σ is an open set.

Accordingly equation (11) is verified, if $\Phi(e)$ is of positive type. But since any completely additive function of e is the difference of two functions of positive type, equation (11) holds for any $\Phi(e)$. It holds, moreover, even if the number of intervals is infinite.

3.1. We shall now use these results in order to show that $\lim\limits_{r=1} U(r, \theta) = 0$ for almost all θ, restricting ourselves first to functions $\Phi(e)$ of positive type.* Suppose that the statement is not true, and that there is therefore a set κ of values of θ and positive numbers m and ϵ such that

$$\text{meas } \kappa > m > 0$$

$$\overline{\lim_{r=1}} \; (U(r, \theta) > \epsilon > 0, \qquad \theta \text{ in } \kappa.$$

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* This is approach to zero "in the narrow sense." The reader may devise an example to show that approach to zero "in the wide sense" (see II, Ch. II) may be impossible at every point of C. Such an example has been constructed by C. H. Dix.

We shall show that this is impossible by slightly extending a result already proved for continuous functions (See III, p. 317).

We take r_0 sufficiently near 1 so that the integral I of (11) is small, $< \epsilon m$. Let E_{r_0} be the set of points (r, θ), $r_0 < r < 1$, for which $U(r, \theta) > \epsilon$; the projection of E_{r_0} on C includes κ and is of measure $> m$. Moreover E_{r_0} is an open set, by Theorem 3, and hence is the sum of a denumerable infinity of curvilinear rectangles, bounded by arcs and radii. On the boundary of each rectangle $U(r, \theta) > \epsilon$, and accordingly, again by Theorem 3, each rectangle is contained in one whose projection on C is an open set, but for every point of which $U(r, \theta) > \epsilon$. The resulting rectangles constitute a family $\mathcal{E}$, the members of which may, of course, overlap.

On the other hand we can take a closed set κ' in κ whose measure is as near that of κ as we please, and thus so that meas. $\kappa' > m$. Each point of κ' will be interior to some arc which is the projection of some one or more of the rectangles of $\mathcal{E}$. Since κ' is a closed set there will be a finite number of these rectangles which have the same property, and thus whose projection on C will constitute a set of measure $> m$.

We may now finally choose a finite number of arcs (θ_i', θ_i'') lying each one interior to one of these rectangles, thus corresponding to values $r > r_0$, but whose projections on C do not overlap and yet have a total measure $> m$. The integral I extended over these arcs will therefore be $> \epsilon m$. But this is a contradiction.

Hence if $\Phi(e)$ is of positive type, $\lim\limits_{r=1} U(r, \theta) = 0$, for almost all θ. But the same result therefore holds if $\Phi(e)$ is any completely additive function of point sets in Σ.

A further result may be obtained. If $\Phi(e)$ is of positive type, so that $U(r, \theta) \geq 0$, we have the two equations

$$\lim_{r=1} \int_0^\theta U(r, \theta)\, d\theta = 0$$

$$\lim_{r=1} U(r, \theta) = 0, \quad \text{almost everywhere.}$$

It follows then, from the well-known theorem of de la Vallée Poussin, that the absolute continuity of $\int U(r, \theta)\, d\theta$ is uniform for any denumerable sequence of values of r, with $\lim r = 1$.

We can prove directly, however, a more general property. Since $U(r, \theta)$

is a potential functoin for its vector derivative we have, by integrating over the region $r_1 < r < r_2$, $\theta' < \theta < \theta''$, which we denote by $\sigma(r_1, r_2)$,

$$\int_{\sigma(r_1,r_2)} (1/r) D_r U \, d\omega = \int_{\theta'}^{\theta''} U(r_2, \theta) d\theta - \int_{\theta'}^{\theta''} U(r_1, \theta) d\theta,$$

so that if $\Phi(e)$ is of positive type,

$$\int_{\theta'}^{\theta''} U(r_2, \theta) d\theta \leq \int_{\theta'}^{\theta''} U(r_1, \theta) d\theta + \int_{\sigma(r_1,1)} (1/r) \mid D_r U \mid d\omega$$

$$\leq \int_{\theta'}^{\theta''} U(r_1, \theta) d\theta + \int_{\theta'}^{\theta''} d\theta \int_{r_1}^{1} \mid D_r U \mid dr$$

since the double integral of the right-hand member may be written as an iterated integral.

But both terms of the right-hand member now define absolutely continuous functions of θ which are independent of r_2. Hence the absolute continuity of the function $\int_0^\theta U(r_2, \theta) d\theta$ is uniform for all r_2, $r_1 < r_2 < 1$. The same result holds accordingly even if $\Phi(e)$ is not of positive type.

The results of this section may be expressed in the following theorem:

THEOREM 4. *The principal solution $U(r, \theta)$, for the unit circle C, given by* (4), *satisfies the relations*

(12) $$\lim_{r=1} \int_0^\theta U(r, \theta) d\theta = 0$$

(13) $$\lim_{r=1} U(r, \theta) = 0, \quad \textit{for almost all } \theta,$$

(14) $\int_0^\theta U(r, \theta) d\theta$ *is absolutely continuous as a function of* θ, *uniformly in* r, *for* $0 < r_1 < r < 1$,

(15) $\int_0^{2\pi} \mid U(r, \theta) \mid d\theta$ *is bounded,* $0 < r_1 < r < 1$.

Equation (14) is the condition (ii), already mentioned, and (15) is the condition (i), which of course is a consequence of (ii).

4. *Discontinuous first boundary value problems for the circle.* If two solutions of (2) are taken as the same when they differ at most on sets of points of zero superficial measure, the following theorem may be stated.

THEOREM 5. *Consider the class of functions $u(r,\theta)$ which are potential functions of their vector derivatives and satisfy Poisson's equation* (2), *for almost all segments in the unit circle Σ (or almost all circles in Σ). Let $F(\theta)$ be of limited variation, with regular discontinuities and such that $F(\theta+2\pi) = F(\theta)+F(2\pi)-F(0)$; and let $f(\theta)$ be summable in the Lebesgue sense over the closed interval $(0,2\pi)$.*

There is one and only one $u(r,\theta)$ which satisfies (i) [i. e., equation (15)] and the boundary condition

$$\lim_{r=1}\int_0^\theta u(r,\theta)\,d\theta = F(\theta)-F(0). \tag{16}$$

This solution takes on the boundary values $F'(\theta)$ almost everywhere, for approach along a radius.

There is one and only one $u(r,\theta)$ which satisfies (ii) [i. e., equation (14)] *and takes on the boundary values $f(\theta)$, almost everywhere, for approach along a radius.*

These functions in both cases will satisfy (2) for almost all curves of class Γ.

In fact, as already remarked in § 2. 3, the difference $u(r,\theta)-U(r,\theta)$ will be, except for removable discontinuities on at most a set of zero superficial measure, continuous with all its derivatives and a solution of Laplace's equation in Σ. This difference will then be uniquely determined, if it is of the class (i), by the boundary values of its integral, and if it is of the class (ii), by its own boundary values taken on almost everywhere, approaching the boundary along radii, that is, in the *narrow* sense (see II, Ch. III). But since $U(r,\theta)$ satisfies both (i) and (ii), $u(r,\theta)-U(r,\theta)$ satisfies whichever of these conditions is satisfied by $u(r,\theta)$; moreover the boundary condition on $u(r,\theta)-U(r,\theta)$ is the same as that on $u(r,\theta)$, by (12) and (13) of Theorem 4.

The requirement that (2) be satisfied for almost all segments or almost all circles in Σ may be modified by substituting for these other special classes of closed curves, e. g., almost all cells of a translatable regular net of curves of class Γ (see IV, p. 126).

5. *The general bounded simply connected plane region T.* Let now $g(Q,M)$ be again the Green's function for T with pole Q, and $h(Q,M)$ its

conjugate. The conditions (i) and (ii) take a form which is invariant of a conformal transformation:

(*i*) *The integral*

$$\int_0^{2\pi} |\, u(M) \,|\, dh(Q,M) = \int_{g=x} |\, u(M) \,| \, \frac{\partial g(Q,M)}{\partial n_M} \, ds_M$$

extended over the closed curve $g(Q,M) = x$ *remains bounded as* x *tends to zero.*

(*ii*) *The integral*

$$\int_0^h u(M)\,dh(Q,M),$$

extended along the curve $g(Q,M) = x$ *is absolutely continuous in* h, *uniformly with respect to* x, *as* x *tends to zero.*

We consider *a class* (A) of functions $u(M)$ which are potential functions of their vector derivatives and solutions of (2) for almost all segments (or almost all circles, or other special classes of curves, as already mentioned, or almost all curves of class Γ). We let $F(P) = F(h)$ *be of limited variation on the accessible boundary points of* T (see II, p. 78), with regular discontinuities and such that $F(h + 2\pi) = F(h) + F(2\pi) - F(0)$, and we let $f(P) = f(h)$ be summable in the Lebesgue sense on the boundary (see II, p. 78) with respect to $h(Q,P)$. We state the theorem:

THEOREM 6. *There is one and only one* $u(M)$ *of class* (A) *which satisfies* (*i*) *and the boundary condition*

$$\lim_{x=0} \int_0^h u(M)\,dh(Q,M) = F(P) - F(P_0) \tag{17}$$

for points of the boundary accessible along curves $h = const.$ *from* Q, *where* $F(P_0)$ *is the value of* $F(P)$ *which corresponds to* $h = 0$ *and the integral is extended along curves* $g(Q,M) = x$.

This solution takes on the boundary values dF/dh *almost everywhere, in the narrow sense, that is, along curves* $h =$ const., *and satisfies* (2) *for almost all curves of class* Γ.

There is one and only one $u(M)$ *of class* (A) *which satisfies* (*ii*) *and the boundary condition*

(18) $$\lim_{M=P} u(M) = f(P),$$ *in the narrow sense, almost everywhere on the boundary of T.*

This solution satisfies (2) *for almost all curves of class* Γ.

The meanings assigned to $F(P)$ and $f(P)$, almost all, etc., depend upon a definition of the order of boundary points (see II, Ch. V). The $f(P)$ is independent of the pole Q, but the $F(P)$ depends on Q, for the same function $u(M)$, according to the equation

$$F_Q(P) - F_Q(P_0) = \int_0^h \frac{dh(Q,P')}{dh(Q',P')}\, dF_{Q'}(P').$$

Certain continuous boundary value problems occur as special cases of Theorem 6, precisely as in the case of Laplace's equation (see II, Ch. V). A function $u(M)$ which happens to remain bounded in the neighborhood of the boundary of T of course satisfies (ii).

5.1. The theorem will follow immediately from the similar theorems on Laplace's equation (see II, Ch. V), by showing merely that the principal solution $U(M)$ for T satisfies (i) and (ii) and takes on zero boundary values in the sense of (17) and (18); for the difference $u(M) - U(M)$ will then, except for removable discontinuities, be a solution of Laplace's equation. The proof will be accomplished by means of a conformal transformation of T into the interior Σ of the unit circle. In fact, considerably more of the general problem might have been handled by this method, since the quantity

$$\int_w D_n u \, ds$$

is invariant of a conformal transformation, provided the curves are such as to give the integral meaning both in T and Σ; but the treatment is briefer in the present fashion.

It will evidently be sufficient to consider $\Phi(e)$ of positive type.

Let us make then a transformation of T into Σ by means of which Q passes into O and $h = 0$ into $\theta = 0$, the function $-\theta$ being conjugate to $\log 1/r$. We define a function $\Psi(e')$ in Σ by writing

$$\Psi(e') = \Phi(e),$$

where e is the set in T which transforms into e' by the one-to-one correspondence. The function $\Psi(e')$ will then also be of positive type and bounded in Σ.

Let $\{e_i'\}$ denote a denumerable infinity of sets in Σ, no two of them with common points, and let $\{e_i\}$ be the aggregate of corresponding sets in T. No two of the latter have common points. The set $E = e_1 + e_2 + \cdots$ will then correspond to the set $E' = e_1' + e_2' + \cdots$; for if a point M' is in E' it will be a member of one of the sets e_i' and its corresponding point M will lie in e_i and therefore in E; also, similarly, if M lies in E its transform M' will lie in E'. It follows that $\Psi(e')$ is completely additive. In fact, with the notation just used,

$$\sum_1^n \Psi(e_i') = \sum_1^n \Phi(e_i)$$

and

$$\lim_{n=\infty} \sum_1^n \Psi(e_i') = \lim_{n=\infty} \sum_1^n \Phi(e_i) = \Phi(E).$$

But, by definition,

$$\Psi(E') = \Phi(E)$$

and therefore

$$\Psi(E') = \lim_{n=\infty} \sum_1^n \Psi(e_i').$$

Moreover, if we denote by $G(M', P')$ and $g(M, P)$ the respective Green's functions in the two regions Σ and T, we shall have

$$(19) \qquad \int_\Sigma G(M', P') d\Psi(e'_{P'}) = \int_T g(M, P) d\Phi(e_P).$$

In order to prove this identity, we denote by Σ_x and T_x respectively the open regions bounded by the curves $G(M', P') = x$ and $g(M, P) = x$, and form for $n > x$ the functions $H_n'(P')$ and $H_n(P)$:

$$\begin{aligned} H_n(P) &= g(M, P) \quad &, \quad & g(M, P) < n, \\ &= n &, \quad & g(M, P) \geq n, \\ H_n'(P') &= G(M', P') &, \quad & G(M', P) < n, \\ &= n &, \quad & G(M', P') \geq n. \end{aligned}$$

Then

$$H_n'(P') = H_n(P)$$

and

$$\int_{\Sigma_x} H_n'(P')\, d\Psi(e') = \int_{T_x} H_n(P)\, d\Phi(e)$$

since both may be written as limits of the same Riemann sum.

But if we let x approach zero, we have

$$\int_{\Sigma} H_n'(P')\, d\Psi(e') = \int_T H_n(P)\, d\Phi(e)$$

since

$$\int_T H_n(P) d\Phi(e) - \int_{T_x} H_n(P) d\Phi(e) = \int_{T-T_x} H_n(P) d\Phi(e)$$
$$\leq x\Phi(T - T_x),$$

with a similar inequality for the left-hand member. If now we let n become infinite we have the result desired, from the definition of the *general integral.*

Equation (19) states that the fundamental solutions $U'(M')$ in Σ and $U(M)$ in T are equal for corresponding points M' and M. We accordingly conclude that, in T, $U(M)$ satisfies (i) and (ii), and also the equations

$$\text{(17')} \qquad \lim_{x=0} \int_0^h U(M) dh(Q, M) = 0,$$

where the integral is extended along the curve $g(Q, M) = x$, and

$$\text{(18')} \qquad \lim_{M=P} \quad U(M) = 0,$$

almost everywhere, in the narrow sense. This completes the proof of Theorem 6.

It is worthy of notice that if we transform the function of curves $\Phi(w)$ as we transform the function of point sets $\Phi(e)$, the regularity of the discontinuities is preserved. That is, if we denote by w' the curve in Σ which corresponds to the regular curve w in T, w' will be regular, and if we write $\Psi(w') = \Phi(w)$, the equation

$$\Psi(w') = \int_{\Sigma} q(w', P')\, d\Psi(e')$$

will hold for $\Psi(w')$ in Σ' if the corresponding equation (1) holds in T. In fact, since the transformation is conformal the density function of w at P in T becomes the density function of w' at P' in Σ:

$$q(w, P) = q(w', P'),$$

and hence, as may easily be proved,

$$\int_{\Sigma} q(w', P')\, d\Psi(e') = \int_T q(w, P)\, d\Phi(e).$$

It is in precisely this sense that (2) is invariant of a conformal transformation.

6. *The problem for three dimensions.* There is no essential difficulty

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in generalizing the results of §§ 1 and 2 to three dimensions; and §§ 3 and 4 may evidently be followed in an analysis of the sphere. But we cannot say the same for § 5, since a transformation of the kind there used is not available in three dimensions. Results similar to those here expounded for the general plane region may however be obtained for space regions bounded by surfaces which are sufficiently smooth, and the integration with respect to the conjugate of the Green's function in the conditions (i) and (ii) may be replaced by integration with respect to the element of area over surfaces close to the boundary surface. Corresponding simplifications occur with sufficiently smooth boundaries in the plane. But further consideration of these situations may well be postponed until corresponding treatments of the Dirichlet problem, now in preparation, are available.

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On Irreducible Cross-cuts of Plane Simply Connected Regions.*

By Wallace Alvin Wilson.

1. It is a well-known theorem that, if R is a simply connected plane region whose frontier F is bounded and C is a simple arc lying in R save for the end-points, which are on F, then C cuts R into two simply connected regions and the frontier of each of these contains C. Moreover, if m is a point of one of these regions and n one of the other, no closed proper part of C separates m from n in R. These properties are obviously not confined to simple arcs and it is the purpose of this paper to extend these results to a more general class of continua.

It is an easy guess that the generalization of the simple arc C will be a bounded continuum irreducible between two or more sets of points and at first sight it seems that most properties obtainable would be obvious corollaries of certain theorems already obtained by C. Kuratowski † and the author ‡ regarding regular frontiers (i. e., frontiers which are the common frontiers of at least two components of their complements). However, there are two complications: first, the frontier F may not be regular; and second, F may itself lie in different components of the complement of C. As a simple example let F be the union of the segments ab and cd, where in terms of Cartesian co-ordinates $a = (-2, 0)$, $b = (3, 0)$, $c = (1, 0)$, and $d = (0, 1)$. Let M and N be circumferences of radius 1 and centers $(-2, 0)$ and $(2, 0)$, respectively, and let C be the union of M and N and an open curve approaching M and N asymptotically, but not meeting $M + N + F$. Then C is irreducible between the set $b + c$ and the point $(-1, 0)$ and divides R, the complement of F, into five simply connected regions, two having C as a proper part of their frontiers. But F is not regular and lies in three components of the complement of C.

The work falls naturally into three parts: §§ 2-7, containing definitions

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† (A) C. Kuratowski, "Sur les coupures du plan," *Fundamenta Mathematicae*, Vol. 6, pp. 130-145.

‡ (B) W. A. Wilson, "On Bounded Regular Frontiers in the Plane," *Bulletin of the American Mathematical Society*, Vol. 34, pp. 81-90.

and general properties; §§ 8-10, containing some useful lemmas; and §§ 11-14, containing the principal theorems.

2. *Definitions.* In the following let R be a simply connected plane region whose frontier F is bounded. If C is a sub-set of $\bar{R}$, m and n are two points of R not on C, and every continuum contained in R and joining m and n meets C, we say that C is a *cross-cut of* R *between* m *and* n and we write that C is an $S(m, n, R)$. If C is closed and no closed proper part of C is an $S(m, n, R)$, we say that C is an irreducible $S(m, n, R)$; if C is an $S(m, n, R)$, but is not itself an $S(m, n)$, it will be called *regular*. It is easily seen that, if C is a bounded closed $S(m, n, R)$, it contains an irreducible $S(m, n, R)$.

Examples. Let R be the interior of a circumference F and let C be a circumference in $\bar{R}$ tangent to F at some point. If m lies within and n without C, C is a non-regular irreducible $S(m, n, R)$.

Let R be as before and let C be the union of F and a double spiral starting at the center of R and approaching F asymptotically. Then C cuts R into two regions and is a non-regular irreducible $S(m, n, R)$. Here $C \supset F$.

Let R be as before and let C be the union of a circumference in R and two segments joining this circumference to F. Here C is a non-regular $S(m, n, R)$, and is not irreducible.

3. *Notation.* Aside from the ordinary notation of the aggregate theory, the following will be used.

The whole plane will be denoted by Z.

If F is a continuum, a component of $Z - F$ whose frontier is F is called a *principal* component of $Z - F$.

The symbol $V_\delta(a)$ denotes those points of the plane whose distances from the point a are less than δ.

If every continuum in the plane which joins the points m and n meets C, we say that C is an $S(m, n)$.

The notation "$A \subseteq B$" signifies that the set A is a part of the set B and may be identical with it.

4. THEOREM. *Let R be a simply connected region whose frontier F is bounded and let C be a bounded regular irreducible $S(m, n, R)$. Then $F \cdot C$ is not connected.*

Proof. As neither F nor C is an $S(m, n)$, $F + C$ is not an $S(m, n)$

if $F \cdot C$ is a continuum, by a theorem essentially due to Z. Janiszewski.* But $F + C$ is an $S(m, n)$, since any arc mn not meeting F lies in R and hence meets C.

5. THEOREM. *Let R be a simply connected region whose frontier F is bounded. For the bounded closed set C to be an irreducible $S(m, n, R)$ it is necessary and sufficient that m and n lie in components R_m and R_n of $R - R \cdot C$ whose frontiers both contain C and that $C = \overline{R \cdot C}$.*

This theorem and the following are easy generalizations of similar theorems by C. Kuratowski [See reference (A), pp. 133 and 136]. The proof of the above is as follows.

Let C be an irreducible $S(m, n, R)$. It is evident that $R \cdot C$ is an $S(m, n, R)$ and that $\overline{R \cdot C}$ is a closed $S(m, n, R)$. Hence $C = \overline{R \cdot C}$. Clearly m and n lie in different components R_m and R_n of $R - R \cdot C$. Let H be the frontier of R_m and $H \cdot C \neq C$. Any continuum in R joining m and n must cut H. It does not cut F and hence must cut $H \cdot C$, since $H \subset F + C$. Thus $H \cdot C$ is an $S(m, n, R)$. Therefore $H \cdot C = C$.

Conversely, suppose that C is a part of the frontiers of both R_m and R_n and that $C = \overline{R \cdot C}$. Obviously C is an $S(m, n, R)$. Since it is bounded and closed, there is a closed sub-set D which is an irreducible $S(m, n, R)$. If $D \neq C$, there is a point x of $C - D$ which lies in R. Then there is a $\delta > 0$ such that $V_\delta(x) \subset R$ and $V_\delta(x)$ contains points of both R_m and R_n, which contradicts the hypothesis regarding D. Hence C is an irreducible $S(m, n, R)$.

6. THEOREM. *Let R be a simply connected region whose frontier F is bounded. Let C be a bounded irreducible $S(m, n, R)$. Then C is a continuum and $C^* = R \cdot C$ is connected.*

Proof. If C^* is not connected, $C^* = M + N$, where $\bar{M} \cdot N = M \cdot \bar{N} = 0$ and neither M nor N is void. Let R_m and R_n be the components of $R - R \cdot C$ containing m and n, respectively. If x is a point of M, there is a $V_\delta(x)$ containing no point of $N + F$, but containing points of both R_m and R_n. Hence $N + F$ is not an $S(m, n)$, and the same thing is true of $M + F$. But both of these sets are closed and $(N + F)(M + F) = F$. This is a contradiction by reference (C). Hence C^* is connected. As $C = \overline{C^*}$ by § 5, C is a continuum.

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* (C) Z. Janiszewski, "Sur les coupures du plan faites par des continus," *Prace Matem.-fizyczne*, Vol. 26. Theorem A: If P and Q are bounded closed sets, if $P \cdot Q$ is connected, and neither P nor Q is an $S(m, n)$, then $P + Q$ is not an $S(m, n)$. See also S. Straszewicz, *Fundamenta Mathematicae*, Vol. 4, p. 129.

7. If for each pair of points $\{a_i\}$, $i = 1, 2, \cdots, n$, C is a bounded irreducible $S(a_i, a_j, R)$, it is convenient to write that C is an irreducible $S(a_i; R)$. The previous theorems are of course valid in this case. In particular, it follows that there are n components of $R - R \cdot C$ such that C is a part of the frontier of each one.

As an example of an irreducible $S(a_i; R)$, $i = 1, 2, \cdots, k$, we need only to modify the construction of the continuum H described in reference (B), pp. 88-90. Let F be the segment ab. It is apparent from the discussion there given that the construction can be made so that there are k simply connected regions analogous to $E_1 + G_4 + \cdots + G_{3n-2} + \cdots$ and that the frontier of each of these is the union of H and a segment of ab. If we take $C = H$, $R = Z - F = Z - ab$, and the points $\{a_i\}$ one in each of the k connected regions just described, it is evident by § 5 that C is an irreducible $S(a_i; R)$.

8. Before proceeding further it is desirable to consider some general properties of the union of two bounded continua whose divisor is not connected and of their complementary domains. Let H and K be the continua in question, $F = H + K$, $H^* = F - K$, $K^* = F - H$, and let α be any component of $H \cdot K$. If Z denotes the plane, the components of $Z - H$ and $Z - K$ containing no points of K^* and H^*, respectively, fall into three classes: first, components $\{R_h\}$ whose frontiers are parts of H but not of K; second, components $\{R_k\}$ whose frontiers are parts of K but not of H; third, components $\{R_{hk}\}$ whose frontiers are parts of $H \cdot K$. No component of one class has a point in common with one of another and each one is a component of $Z - F$. Also the frontier of each R_{hk} is a part of some component of $H \cdot K$.

The components of $Z - F$ fall into four classes: first, those having frontier points on both H^* and K^*, second, those having frontier points on H^* but not on K^*; third, those having frontier points on K^* but not on H^*; fourth, those having frontier points on neither H^* nor K^*. It is easily shown that the components of the second, third, and fourth classes are identical with the components $\{R_h\}$, $\{R_k\}$, and $\{R_{hk}\}$, respectively, discussed above. Furthermore, by an extension of a theorem by S. Straszewicz * it can be shown that, if $H \cdot K$ has n components (or an infinity), there are at least n components of $Z - F$ of the first class (or an infinity).

If we set $H_1 = H + \sum R_h + \sum R_{hk}$ and $K_1 = K + \sum R_k + \sum R_{hk}$, we

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* See reference (B), p. 83 and S. Straszewicz, "Über die Zerschneidung der Ebene," *Fundamenta Mathematicae*, Vol. 7, p. 174.

obtain the following results. Each component of $H_1 \cdot K_1$ is a component α of $H \cdot K$ together with such components $\{R_{hk}\}$ as have their frontiers a part of α. If H^* is connected, so is H^*_1. If $F_1 = H_1 + K_1$, the components of $Z - F_1$ are identical with those of $Z - F$ of the first class.

9. LEMMA. *Let H and K be bounded continua. Let $H^* = H - H \cdot K$ be connected and $\overline{H^*} = H$. Then all the points of K lying in any component R of $Z - H$ are on a component of $K \cdot \bar{R}$.*

Proof. Let us assume the existence of two points m and n in $R \cdot K$ such that any sub-continuum M of K irreducible between m and n contains points not in $\bar{R}$. Let C be a simple arc in R whose ends are m and n. Then $C \cdot M$ is a closed set and $C \cdot M \neq C$. Hence $C - C \cdot M$ is a finite or enumerable set of open arcs whose end-points lie in M. There is at least one of them, with end-points m' and n', such that no sub-continuum M' of M irreducible between m' and n' lies in $\bar{R}$; let it together with its end-points be denoted by C'. This last statement holds, as otherwise the hypothesis regarding M would be false.

As $C' \cdot M' = m' + n'$, there are exactly two principal components S and T of $Z - (C' + M')$, by reference (B), p. 86. As $C' \subset R$, $S \cdot R \neq 0$ and $T \cdot R \neq 0$. As M' contains points not in $\bar{R}$, $S \cdot (Z - \bar{R}) \neq 0$ and $T \cdot (Z - \bar{R}) \neq 0$. Thus both S and T contain points of H, as they connect points in R and points not in $\bar{R}$. Since $H^* \cdot (C' + M') = 0$ and H^* is connected, H^* lies in one component of $Z - (C' + M')$. Suppose that $S \cdot H^* = 0$. Then $S \cdot \overline{H^*} = S \cdot H = 0$, which is a contradiction.

The theorem follows at once. A consequence is that a sub-continuum of K joining any point of $K \cdot \bar{R}$ not on the component containing $R \cdot K$ to this component must contain points not in $\bar{R}$.

10. By means of a device employed by R. L. Moore in his paper "On the Separation of the Plane by a Continuum" (*Bulletin of the American Mathematical Society,* Vol. 35, pp. 303-307), there can be obtained certain separation properties embodied in the following lemma and corollaries. The term "upper semi-continuous image" used in the lemma signifies that there is a one to one correspondence between the elements $\{X\}$ of J and the points $\{t\}$ of a circumference, in which $X = f(t)$ is upper semi-continuous. The properties of such relations are discussed in an article by Moore ("Concerning Upper Semi-Continuous Collections of Continua," *Transactions of the American Mathematical Society,* Vol. 27, pp. 416-428), one by L. S. Hill ("Properties of Certain Aggregate Functions," this *Journal,* Vol. 49, pp. 419-432),

and one by C. Kuratowski ("Sur les decompositions semi-continues d'espaces metriques compacts," *Fundamenta Mathematicae,* Vol. 11, pp. 170-185).

Lemma. *Let H and K be bounded continua and $H \cdot K$ be not connected. Let $H^* = H - H \cdot K$ be connected. Let $K^* = K - H \cdot K$ lie in one component of $Z - H$. Then there is a bounded continuum J having these properties: $J \supset H \cdot K$; $Z - J$ has two components, one containing H^* and the other K^*; J is the upper semi-continuous image of a circumference; each element of J is a point not on $H + K$ or a component α of $H \cdot K$ together with such components of $Z - (H + K)$ as have their frontiers on α.*

*Proof.** At least one component of $Z - (H + K)$ has frontier points on both H^* and K^*. By inversion there is no loss in generality in assuming this unbounded.

Let H_1 and K_1 be as defined in § 8. Then the continua H_1 and K_1 are bounded by the previous paragraph and neither cuts the plane by our hypotheses regarding H^* and K^*. Also H_1^* is connected. $H_1^* \supseteq H^*$, and $K_1^* \supseteq K^*$.

If each component of $H_1 \cdot K_1$ is regarded as an element and each point of $Z - H_1 \cdot K_1$ as an element, this set of elements is a space S topologically equivalent to the plane, by Moore's paper in the *Transactions* referred to above. Let the images in S of sets in Z be denoted by the corresponding small letters. Then the set $h_1 \cdot k_1$ is totally disconnected and so h_1 and k_1 satisfy the hypothesis of another theorem by Moore.† There is then a simple closed curve j in S such that $j \supset h_1 \cdot k_1$ and h_1^* and k_1^* are in different components of $S - j$. Returning to the plane, we see that H_1^* and K_1^*, and a fortiori H^* and K^*, lie in different components of $Z - J$.

As j is a simple closed curve in S, J is the upper semi-continuous image of a circumference. Its construction insures that the elements of J are as described in the statement of the lemma.

Corollary 1. *The points of $J - H_1 \cdot K_1$ are common frontier points of*

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* Since this article was submitted to the editors, there has appeared an article by C. Kuratowski ("Sur la separation d'ensembles situes sur le plan," *Fundamenta Mathematicae,* Vol. 12, pp. 214-239) which contains an interesting discussion of the subject of separation of sets by continua of the nature of J. By virtue of § 9 this lemma can be deduced as a corollary of Theorem V by Kuratowski. The proof here given has been retained as being perhaps of interest to those familiar with Moore's earlier work along the same line.

† R. L. Moore, "Concerning the Separation of Point Sets by Curves," *Proceedings of the National Academy of Sciences,* Vol. 11, p. 470.

the two components of $Z - J$ and they all lie in the component of $Z - H$ containing K^*.

For, if x is such a point, there is a $\delta > 0$ so small that all the points of J in $V_\delta(x)$ lie on an arc of J composed wholly of elements which are points. Since J is a simple closed curve of elements, the part of J not within this arc does not separate Z. Hence x is a frontier point of both components of $Z - J$. Now, if two points of $J - H_1 \cdot K_1$ are in different components of $Z - H$, there are two arcs L_1 and L_2 of J whose elements are points and which lie in different components of $Z - H$. Then, if I is the component of $Z - J$ containing H^*, L_1 and L_2 can be joined in the other component E of $Z - J$, without crossing H, a manifest contradiction.

COROLLARY 2. *If R is a component of $Z - (H + K)$ which has frontier points on H^* and K^*, then $J' = J \cdot R$ is the homeomorphic image of a simple open arc and $\overline{J'} - J'$ consists of two continua lying on different components of $H \cdot K$.*

Proof. If J' is not connected, there is in R a simple arc mn such that $J' \cdot (mn) = m + n$ and m and n lie on different components of J'. From the construction of J, m and n lie on simple arcs A_1 and A_2, respectively, of J, which lie in R, and $\overline{J - (A_1 + A_2)} = B_1 + B_2$, where B_1 and B_2 are continua, $B_1 \cdot B_2 = 0$, and $H \cdot K \subset B_1 + B_2$. Also both B_1 and B_2 contain points of $H \cdot K$, for otherwise one of them, say B_1, does not meet $H + K$. Then $B_1 \subset R$ and m and n would be on the same component of J'.

Let I and E be the components of $Z - J$ and $mn - (m + n) \subset E$. Then mn divides E into two simply connected regions E_1 and E_2, where the frontier of each contains points of one of the sets B_1 and B_2, but not the other. If $E \supset K^*$, we have a contradiction. For $\bar{E}_1 + B_1 + A_1 + A_2$ and $\bar{E}_2 + B_2 + A_1 + A_2$ are continua both of which contain points of the continuum K, their common part is $mn + A_1 + A_2$, and $K \cdot (mn + A_1 + A_2) = 0$. Likewise, we have a contradiction if $E \supset H^*$. Thus J' is connected.

Since J' is connected and each of its elements is a point and since J is the upper semi-continuous image of a circumference, J' is the homeomorphic image of a simple arc C. This arc is open. To prove this set $B = J - J'$. Neither B nor J' is an $S(I, E)$. If C were closed, J' would be a continuum and $B \cdot J' = 0$, which is impossible. If C were closed at one end, $B \cdot \overline{J'} = \overline{J'} - J'$ would be a continuum, which is a contradiction by reference (C).

The two continua making up $\overline{J'} - J'$ lie on two elements M and N of B. If $M = N$, we have the same contradiction as when $\overline{J'} - J'$ is a con-

tinuum. But $\overline{J'} - J'$ contains no inner points of M or N. Hence $\overline{J'} - J'$ lies on two components of $H \cdot K$.

11. THEOREM. *Let R be a simply connected region whose frontier K is bounded. Let H be a bounded regular irreducible $S(a_i; R)$, $i = 1, 2, \cdots, n$. Then $H \cdot K$ is the sum of n or more closed sets $\{\alpha_i\}$ between each pair of which H is an irreducible continuum.*

Proof. Since H is regular, all the components $\{R_i\}$ of $Z - (H + K)$ containing points $\{a_i\}$ lie in one component S_1 of $Z - H$. By § 9, if $K_1^* = K \cdot S_1$, there is a component K_1 of $K \cdot \bar{S}_1$ containing K_1^* and $K_1^* = K_1 - H \cdot K_1$. Now let $M = K - K_1^*$ and set $H_1 = H + M$. Since K is a continuum, so is H_1. It is evident that $H_1 + K = H + K$, S_1 is a component of $Z - H_1$, and $K_1^* = K - H_1 \cdot K$. If we set $H_1^* = H_1 - H_1 \cdot K$, $H_1^* = H^*$. By §§ 5 and 6, H^* is connected and $\overline{H^*} = H$. Moreover, neither $H + M$ nor K is an $S(a_i, a_j)$, while $H + K$ is. Hence, by reference (C), $H_1 \cdot K$ is not connected. Thus H_1 and K satisfy all hypotheses of the lemma of § 10 and there is a continuum J with the properties there given.

By § 10, Corollary 1, all the elements of J which are points of $J - H_1 \cdot K$ lie in S_1. Since each R_i has frontier points on both H^* and K^*, as H is regular, and since $R_i \subset S_1$, R_i has frontier points on both H^* and K^*_1. Hence each R_i contains an arc A_i of J whose elements are points and we can write $J = \sum_1^n A_i + \sum_1^n N_i$, where each N_i is a continuum and $N_i \cdot \overline{(J - N_i)}$ consists of an end-point of A_i and one of A_p $(p = i + 1, \text{ mod. } n)$. Now $H \cdot K \subset \sum N_i$ and each N_i contains points of $H \cdot K$, as otherwise $H + K$ would not be an $S(a_i, a_{i+1})$.

Now set $B_1 = N_1$ and $B_2 = \sum_3^n A_i + \sum_2^n N_i$. Let I and E be the components of $Z - J$ containing H^* and K_1^*, respectively. Let A_1^* and A_2^* denote the arcs A_1 and A_2 without their end-points. If H' is a proper closed part of H, there is a point x of H^* not in H'. For some $\delta > 0$ there is a $V_\delta(x)$ in I such that $H' \cdot V_\delta(x) = 0$. Let m be a point in $R_1 \cdot V_\delta(x)$ and n one in $R_2 \cdot V_\delta(x)$. As n lies in I, A_2^* is an open arc, and $K \cdot I = 0$, there is a simple arc nz, such that z lies in A_2^*, $nz - z \subset I$, and $(nz) \cdot (B_1 + A_1 + B_2 + K) = 0$. Since $n + z \subset R_2$, $H + K$ is not an $S(n, z)$. As $(H + K) \cdot (B_1 + A_1 + B_2 + K) = K$, a continuum, $H + K + B_1 + A_1 + B_2$ is not an $S(n, z)$. Then there is an arc nz not meeting any of these sets such that $(nz) \cdot A_2^* = z$ and $nz - z \subset I$.

By § 10, Corollary 1, $A_2 + B_1 + B_2$ is not an $S(H^*, K_1^*)$ and so m can be joined to K_1^* by a simple arc not cutting $A_2 + B_1 + B_2$. Starting

from m, let u be the first point where this meets K_1^* and let w lie on mu and be in some $V_\eta(u)$ so small that $J \cdot V_\eta(u) = 0$. As w is in E, $(mw) \cdot A_1^* \neq 0$. Let mw meet A_1^* for the last time at m'. Then $m'w - m' \subset E$ and $m'w \subset R_1$, since $A_1 \subset R_1$ and $(m'w) \cdot (H + K) = 0$. That is, w lies in R_1. Thus $A_2 + B_1 + B_2 + K$ is not an $S(m, w)$; nor is $H + K$, since $m + w \subset R_1$. Hence $A_2 + B_1 + B_2 + H + K$ is not an $S(m, w)$ and there is an arc mw not cutting any of these sets. Starting from m, let y be the first point where it meets A_1^*. Then $my \subset I + y$ and $(my) \cdot H' = 0$.

Let mn be an arc in $V_\delta(x)$. Then $(mn) \cdot H' = 0$, and $yz = my + mn + nz$ is a simple arc in $I + y + z$ not meeting H' and meeting J only in the points y and z of A_1^* and A_2^*, respectively.

Let H'' be a sub-continuum of H irreducible between B_1 and B_2. If yz is any simple arc in $I + y + z$, where y lies in A_1^* and z in A_2^*, it divides I into two simply connected regions such that the frontier of each contains points of one of the sets B_1 and B_2, but not the other. As H'' is irreducible between B_1 and B_2, $H'' - H'' \cdot (B_1 + B_2)$ is connected and contains points of both these regions. Hence it meets yz, as $H''\,(A_1 + A_2) = 0$. This result in connection with the preceding paragraph is contradictory unless $H'' = H$, i. e., unless H is irreducible between B_1 and B_2.

Now let $\alpha_i = (H \cdot K) \cdot N_i$; then $\alpha_i \cdot \alpha_j = 0$ if $i \neq j$ and $H \cdot K = \sum_1^n \alpha_i$. Thus H is irreducible between α_i and $\sum_2^n \alpha_i$ and, a fortiori, between α_1 and each of the remaining sets $\{\alpha_i\}$. As B_1 was chosen as any one of the sets N_i, this gives the theorem.

12. THEOREM. *Let R be a simply connected region whose frontier K is bounded. Let H be a bounded regular irreducible $S(a_i; R)$ where the set $\{a_i\}$ is enumerably infinite. Then for every integer n, $H \cdot K$ is the sum of n closed sets between each pair of which H is an irreducible continuum. Also $H \cdot K$ can be expressed as the sum of an infinity of closed sets between each pair of which H is an irreducible continuum.*

Proof. The first statement is a mere corollary of § 11. To prove the second proceed as follows. Let the component of $Z - (H + K)$ containing a_i be R_i. Using the third paragraph of the proof of § 11 divides $H \cdot K$ into two closed sets β_{11} and β_{12}, between which H is irreducible. If we now use the arcs A_1, A_2, and A_3 in a similar manner, it is evident that $H \cdot K$ falls into three sets β_{21}, β_{22}, and β_{23}, of which one is identical with one of the

sets β_{11} and β_{12}, say β_{11}, and the sum of the other two is identical with β_{12}, and that H is irreducible between each pair of these. This process may be continued indefinitely. It obviously generates a dyadic decomposition of $H \cdot K$ into an infinity of closed sets $\{\alpha\}$, no two of which have common points. Each α is the divisor of a sequence $\{\beta_{i,n_i}\}$. If α and α' are any two of these sets and i is sufficiently great, $\alpha \subset \beta_{i,n_i}$ and $\alpha' \subset \beta_{i,n'_i}$, where $n_i \neq n'_i$. Hence H is irreducible between α and α'.

13. THEOREM. *Let $F = H + K$ be the union of two bounded continua. Let $H \cdot K$ be the sum of a finite number n of closed sets $\{\alpha_i\}$ between each pair of which H is irreducible. Then there are at least n components of $Z - F$ whose frontiers contain H as a proper part.*

Proof. Since K is bounded, it contains a sub-continuum K' irreducible between α_1 and $\sum_2^n \alpha_i$. Then $K' - H \cdot K'$ is connected and lies in some component S_1 of $Z - H$. By § 9, if $K^* = K - H \cdot K$, $K^* \cdot S_1$ lies on a component K_1 of $K \cdot \bar{S}_1$ and $K^* \cdot S_1 = K_1 - H \cdot K_1$. Then K_1 contains points of n_1 sets $\{\alpha_i\}$ where $n_1 \geqq 2$. If $n_1 \neq n$, there is in like manner a component S_2 of $Z - H$ such that $K^* \cdot S_2$ lies on a component K_2 of $K \cdot \bar{S}_2$, $K^* \cdot S_2 = K_2 - H \cdot K_2$, and K_2 has points of two or more sets $\{\alpha_i\}$ including α_{n_1+1}. Continuing this process, we obtain k components $\{S_j\}$ of $Z - H$ such that for each j, $K^* \cdot S_j$ lies on a component K_j of $K \cdot S_j$, $K^* \cdot S_j = K_j - H \cdot K_j$, and K_j contains points of n_j sets $\{\alpha_i\}$, where $n_j \geqq 2$ and $\sum n_j \geqq n$.

Now consider any one of these, say K_1. Let $K_1^* = K^* \cdot S_1$. Let $\alpha'_i = K_1 \cdot \alpha_i$ if this set is not void. There are n_1 sets $\{\alpha'_i\}$ and $H \cdot K_1 = \sum \alpha'_i$. Let $H^* = H - H \cdot K_1$. Then H and K_1 satisfy the hypotheses of the lemma of § 10 and there is a continuum J with the properties there described.

To each component of an α'_i corresponds an element of J made up of this component and those components of $Z - (H + K_1)$ bounded by it. If β_i denotes the set of elements of J corresponding to components of α'_i, it is easily seen that β_i is closed. Since J is the upper semi-continuous image of a circumference C, β_i corresponds to a closed set γ_i on C and $C - \sum \gamma_i$ is a set of open arcs. It is easily seen that of these at least n_1 arcs $\{l_i\}$ have their end-points on different sets γ_i. The image L_i of l_i is an open arc of elements of J which are points and L_i has improper limiting points on two different sets β_i and hence on two sets α'_i. As $L_i \cdot \sum \beta_i = 0$, $L_i \cdot (H + K_1) = 0$ and L_i lies in some component R_i of $Z - (H + K_1)$. This is also a component of $Z - (H + K)$. As $J - L_i$ does not separate H^* from K_1^*, it is evident that R_i has frontier points on both H^* and K_1^*. Now L_i is a component of $J - \sum \beta_i$. If $J_i = J \cdot R_i$, $L_i \subseteq J_i$. But, by

§ 10, Corollary 2, J_i is connected and meets no β_i. Hence $J_i = L_i$. Thus the existence of n_1 such arcs $\{L_i\}$ proves the existence of n_1 distinct components $\{R_i\}$ of $Z - (H + K_1)$ which have frontier points on both H^* and K_1^*.

Consider any one of these, as R_1. Let $\bar{L}_1$ meet α'_1 and α'_2; then $\bar{L}_1$ is irreducible between α'_1 and α'_2. So is H and $H \cdot \bar{L}_1 \subseteq \alpha'_1 + \alpha'_2$. Then by reference (B), p. 84 there are at least two principal components M and N of $Z - (H + \bar{L}_1)$. Since K_1^* lies in one component of $Z - J$ and H^* in the other, K_1^* lies in one component of $Z - (H + J)$ and, *a fortiori*, in but one component of $Z - (H + \bar{L}_1)$. Suppose that $M \cdot K_1^* = 0$. Then $M \cdot (H + K_1) = 0$ and $M \subset R_1$, as M has limiting points on L_1, which lies in R_1. As $H \cdot R_1 = 0$ and H is a part of the frontier of M, it is therefore a part of the frontier of R_1. We have therefore shown that there are at least n_1 components of $Z - (H + K)$ lying in S_1 and having H as a proper part of their respective frontiers.

As this holds for every S_j, there are at least $\sum n_j \geqq n$ components of $Z - (H + K)$ whose frontiers contain H and points of K^*.

COROLLARY. *Let $F = H + K$ be the union of two bounded continua. For every integer n let $H \cdot K$ be the sum of n closed sets between each pair of which H is irreducible. Then there is an enumerably infinite set of components of $Z - F$ whose frontiers contain H as a proper part.*

Remarks. The conclusion of the above theorem may be stated as follows by using § 5: *Then H is an irreducible $S(a_i; R)$ where R is the component of $Z - K$ containing $H - H \cdot K$ and the points $\{a_i\}$ lie in R and number at least n.* To see that the converse of the above theorem is false, let H be a circumference and K a straight line segment joining a point within H to one without.

14. The actual existence of continua H and K satisfying the hypothesis of §§ 11-13 can be readily established by modifying Ex. 2 on p. 88 of reference (B), as indicated in § 7. On the other hand, Ex. 1, just preceding this, shows that $H \cdot K$ may be the sum of an infinity of closed sets between each pair of which H is irreducible and yet H need not be an irreducible $S(m, n, R)$ for any two points in R. Finally, it should be noted that H may be a regular irreducible $S(a_i; R)$, $i = 1, 2, \cdots, n$ and be an irreducible $S(a_i; R)$, $i = 1, 2, \cdots, m$, where $m > n$. For H may itself be a regular frontier whose complement has any number of principal components.

Various theorems may be deduced from §§ 11-13 by modifying the hy-

potheses. Two examples are given, of which the first is obtained from § 11, and the second from § 13.

THEOREM. *Let R be a simply connected region whose frontier K is bounded. Let H be a bounded continuum dividing R into two or more regions and be a part of the frontier of each of these. Let $H = \overline{H \cdot R}$ and let H itself separate no two points of R. Then H is irreducible between any pair of components of $H \cdot K$.*

To prove this let m and n be any two points of R, lying in components R_m and R_n, respectively, of $R - R \cdot H$. Then H is a regular irreducible $S(m, n, R)$. By §§ 11 and 12 we know that $H \cdot K$ is the sum of two closed sets between which H is irreducible. Let A and B be components of one of these and H be reducible between them. Then there is a sub-continuum h of H irreducible between A and B, and $H - h$ is connected. But by § 8 there are at least two components S and T of $Z - (h + K)$ whose frontiers contain points of both $h - h \cdot K$ and $K - h \cdot K$; let S be the one containing points of $R \cdot H$ not in h. Then T is a component of $R - R \cdot H$ whose frontier does not contain H, contrary to hypothesis.

THEOREM. *Let $F = H + K$ be the union of two bounded continua. Let $H \cdot K$ have a finite number of components and H be irreducible between each pair of them. Let H not cut the plane. Then H is a part of the frontier of every component of $Z - F$ whose frontier is not a part of K.*

For, if there were a component S of $Z - F$ whose frontier contained points of $H - H \cdot K$, but not the whole of H, there would be a proper sub-set h of H such that $h + K$ separated a point m of S from a point n of some component of $Z - F$ whose frontier contains H. But m and n lie in the same component of $Z - K$, since $H - H \cdot K$ is connected. Then K is not an $S(m, n)$, nor is h, since H does not cut the plane. There is then a contradiction by reference (C) unless h contains a continuum joining two components of $H \cdot K$, which in turn is impossible, since H is irreducible between these components.

NEW HAVEN, CONN.

Number of Abelian Subgroups in Every Prime Power Group.

BY G. A. MILLER.

It is well known that whenever a group of order p^m, p being a prime number, contains an abelian subgroup of order p^3 then the number of its abelian subgroups of this order is of the form $1 + kp$.* This is obviously a special case of the following theorem: *If the order of a group is divisible by* $p^{a(a-1)/2+1}$ *then the number of its abelian subgroups of order* p^a *is always of the form* $1 + kp$. This theorem results almost immediately from the fact that every invariant abelian subgroup whose order does not exceed p^{a-1} which appears in such a group must be contained in an invariant abelian subgroup of larger order. Hence it results that if we count every invariant abelian subgroup of order p^a as many times as the number of its subgroups of order p^{a-1} which are invariant under the entire group and note that each of the latter subgroups appears in a number of the former subgroups which is of the form $1 + kp$, and that this is also the form of the number of the latter subgroups which appear in one of the former, it results that if the number of the latter groups is of the form $1 + kp$ this must also be the form of the number of the former subgroups. Hence the given theorem has been established since the number of the abelian subgroups of order p^3 is of the given form.

From the preceding paragraph it results that when α is given we can always make m sufficiently large so that in every group whose order is divisible by p^m the number of abelian subgroups of order p^a is of the form $1 + kp$. Moreover, it is well known that if a non-abelian group of order p^m contains an abelian subgroup of index p then the number of its subgroups of this index is always exactly $1 + p$. In this case, the number of the abelian subgroups of index p^2 must always be of the form $1 + kp$ since every abelian subgroup of this index must be contained in an abelian subgroup of index p. On the contrary, when a group of order p^m, $m > 7$, does not involve an abelian subgroup of index p but involves abelian subgroups of index p^2 the number of the latter subgroups is not necessarily of the form $1 + kp$. In fact, it is obviously possible to construct such groups which contain exactly two abelian subgroups of index p^2 and have a central of index p^4, for every value of p and every value of $m > 7$. To construct such a group we may start with an

* Miller, Blichfeldt, Dickson, *Finite Groups* (1916), p. 126.

abelian group of order p^{m-2} and of type $(1, 1, 1, \cdots)$, and extend this group by an operator s_1 of order p which has p^2 conjugates under this extended group. The group thus obtained may be extended by an operator s_2 which is commutative with s_1 and gives rise to a commutator subgroup of order p^2 which is found in the central of the preceding group and has only the identity in common with the commutator subgroup to which s_1 gives rise. The resulting group satisfies the given conditions.

From the preceding paragraph it results that the number of the abelian subgroups of index p^2 which appear in a group of order p^m is not necessarily either 0 or of the form $1 + kp$. It is, however, easy to prove that whenever a group of order p^m contains at least one abelian subgroup of index p^2 then it must contain an invariant abelian subgroup of this index. To prove this theorem we note that if H_1 is such an abelian subgroup which is non-invariant then it must have a conjugate H_2 such that each of the two subgroups H_1 and H_2 transforms the other into itself in view of the general theorem that every set of conjugate subgroups, or conjugate operators, of a prime power group involves at least p conjugates such that each of them transforms each of the remaining $p - 1$ into itself. If H_1 and H_2 would generate the entire group then each of these subgroups would be invariant. If they generate a non-abelian subgroup of index p then the number of its abelian subgroups of order p^{m-2} must be of the form $1 + kp$. As this subgroup of index p is invariant under the entire group we have established the following theorem: *If a group of order p^m contains at least one abelian subgroup of index p^2 then it must contain an invariant abelian subgroup of this index.*

It was noted above that whenever $m > 7$ it is possible to construct a group of order p^m which contains exactly 2 abelian subgroups of index p^2. In particular, it is possible in this case to construct a group in which the number of abeilan subgroups of index p^2 is not of the form $1 + kp$. We proceed to prove that this is not possible whenever $m \leqq 7$. For this purpose we shall first prove that if a group G of order p^7 contains an abelian subgroup H_1 of order p^5 then the number of its abelian subgroups of this order is always of the form $1 + kp$. It is evidently only necessary to prove that the number of its invariant abelian subgroups of order p^5 is of this form since the number of its non-invariant abelian subgroups of any order must be divisible by p. In view of the theorem at the end of the preceding paragraph it may be assumed that H_1 is an invariant subgroup of G.

If all the other abelian subgroups of order p^5 contained in G are non-invariant our theorem requires no proof. Hence it may be assumed that G contains two invariant abelian subgroups H_1 and H_2 whose common order

is p^5. These subgroups either generate G or they generate a subgroup of order p^6. If, in the latter case, this subgroup is abelian but G is non-abelian then G contains either only one abelian subgroup of order p^6 and hence the number of its abelian subgroups of order p^5 is of the form $1 + kp$, or G contains exactly $p + 1$ abelian subgroups of order p^6 and hence has a central of order p^5. In this case the number of its abelian subgroups of order p^5 is clearly again of the form $1 + kp$. Hence we may assume for the purpose of proving the theorem under consideration that H_1 and H_2 generate a non-abelian subgroup of order p^6 which involves exactly $p + 1$ abelian subgroups of order p^5. These $p + 1$ subgroups are invariant under G since at least two of them have this property.

When G contains an abelian subgroup of order p^6 the number of its abelian subgroups of order p^5 is obviously of the form $1 + kp$. Hence we shall assume in what follows that no abelian subgroup of order p^6 appears in G but that it contains abelian subgroups of order p^5. If G contains an invariant abelian subgroup of this order in addition to the $p + 1$ considered in the preceding paragraph, and if this involves the cross-cut of these $p + 1$ subgroups then this cross-cut is the central of G and it appears in each of the abelian subgroups of order p^5 which appear in G. The number of these subgroups must therefore again be of the form $1 + kp$. On the other hand, if this additional invariant abelian subgroup does not involve the given cross-cut there will be $p + 1$ abelian subgroups of order p^5 which have for their cross-cut another subgroup of order p^4, and p of these do not appear in the group generated by H_1 and H_2. The central of G in this case is of order p^3. As similar remarks apply to the additional sets of $p + 1$ abelian invariant subgroups of order p^5, in case such sets exist, it results that the number of the invariant abelian subgroups of order p^5 contained in G is of the form $1 + kp$ except possibly in the case when H_1 and H_2 generate G and there are no two invariant abelian subgroups of order p^5 in G whose cross-cut is of order p^4.

It may also be assumed that G contains an invariant abelian subgroup of order p^4 which is not found in an invariant abelian subgroup of the order p^5 since the number of the former is of the form $1 + kp$ according to the general theorem stated in the first paragraph of this article. Hence it results that the central of G is of order p^3 and that every abelian subgroup of order p^5 which is found in G involves this central and that no two of these subgroups have any common operator besides those found in the central of G. Moreover, the central of G and its commutator subgroup must be identical. From this it follows that G involves exactly $p + 1$ invariant abelian subgroups

and the following theorem has been established: *If a group of order p^7 contains an abelian subgroup of order p^5 then the number of its abelian subgroups of this order is of the form $1 + kp$.*

We shall now prove that if a group of order p^m contains an abelian subgroup of order p^4 then the number of its abelian subgroups of this order is always of the form $1 + kp$. From the general theorem stated in the first paragraph it results that this theorem is true whenever $m > 6$. Hence it remains only to prove it when $m = 6$, since it is obviously true when $m = 5$. We may again confine our attention to the invariant abelian subgroups of order p^4. Hence we assume that G is a non-abelian group of order p^6 which contains at least two invariant abelian subgroups H_1, H_2 of order p^4. If every abelian invariant subgroup of order p^3 contained in G were found in an invariant abelian subgroup of order p^4 our theorem would clearly be true since the number of the former is known to be of the form $1 + kp$. Hence we may assume in what follows that G contains an invariant abelian subgroup K of order p^3 which does not appear in H_1.

If H_1 and K generate G their cross-cut is of order p and hence the commutator subgroup of G is of order p. This is impossible since it was assumed that K does not appear in an abelian invariant subgroup of order p^4. Hence it results that H_1 and K generate a group of order p^5 whose central is of order p^2 and whose commutator subgroup coincides with this central. This group of order p^5 contains no abelian subgroup of order p^4 besides H_1. The remaining operators of G cannot be commutative with all the operators of the central of this subgroup since K is not contained in an invariant abelian subgroup of order p^4. As every invariant subgroup of index p^2 in any group must involve the commutator subgroup of this group it results that G contains only one invariant abelian subgroup of order p^4. Hence the following theorem has been established: *If a group of order p^m contains an abelian subgroup of order p^4 then the number of its abelian subgroups of this order is always of the form $1 + kp$.*

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Finite Groups in which Conjugate Operations are Commutative.

By C. Hopkins.

In an article entitled *On Groups in which every two Conjugate Operations are Permutable,* Burnside * developed certain general properties of a group G which is subject to the sole restriction that its conjugate operations be commutative. In particular, he proved that if G is of finite order, it is the direct product of its Sylow subgroups,† and is either an abelian or a metabelian group if no Sylow subgroup is of order 3^m. In the same year Fite published his dissertation *On Metabelian Groups,*‡ in which he defined the class of a group G as the number of terms in the sequence $G', G'', \cdots, 1$, where G' is the group of cogredient isomorphisms of G and each succeeding $G^{[i]}$ is the group of cogredient isomorphism of the preceding one. From this definition it is evident that abelian and metabelian groups form the categories of groups of class 1 and 2, respectively. Moreover, these two categories are included in the category of groups defined by the stipulation that every two conjugate operations be commutative. Hence it appears that a group in which conjugate operations are commutative may be conveniently classified and discussed according to its class. This procedure was adopted by Fite in a later article,‖ in which he asserted that the class of a group of finite order in which every two conjugate operations are commutative can not exceed 5.

In this paper we shall prove that the class of a finite group G in which every two conjugate operations are commutative can not exceed 3; we shall examine in some detail the case where G is of class 3 and has exactly three independent generating operations, and we shall exhibit a method of constructing all groups of this sort.

* Burnside, *Proceedings of the London Mathematical Society,* Vol. 35 (1903), pp. 28-37.

† *Loc. cit.,* p. 34.

‡ Fite, *Transactions of the American Mathematical Society,* Vol. 3 (1902), p. 331 et seq.

§ Fite, *loc. cit.,* p. 348.

‖ Fite, *Mathematische Annalen,* Vol. 67 (1909), pp. 498-510.

1. *General Results.* In what follows the symbol G shall denote a group of order 3^m in which every two conjugate operations are commutative. The assumption that G is of order 3^m involves no loss of generality, since it has been demonstrated by Burnside that if G is of finite order it is the direct product of its Sylow subgroups, and is of class 1 or 2 if no Sylow subgroup is of order 3^m. In what follows we are interested only in groups whose class exceeds 2. If G is of class k it cannot have fewer than k independent generating operations. This follows from the fact that such a group must contain an invariant commutator $P_{1\,2\,\ldots\,k}$ which is generated by the k independent operations $P_1, P_2, \cdots, P_k$.*

If a finite group G of order p^m and of class k whose conjugate operations are commutative has more than k independent generating operations, it must contain a subgroup of class k which has exactly k independent generating operations. To prove this we note that the commutator $P_{1\,2\,\ldots\,k}$ is generated by the k operations $P_1, P_2, \cdots, P_k$, no one of which can be expressed in terms of the remaining $k-1$. [If one of the P's could be expressed as the product of powers of the remaining $k-1$, then $P_{1\,2\,\ldots\,k}$ could be represented as a commutator whose k subscripts would involve only $k-1$ distinct symbols. But such a commutator must be the identity].† Consequently the subgroup $H=\{P_1, P_2, \cdots, P_k\}$ has exactly k independent generating operations and is of class k, since its kth, but not its $(k-1)$st, commutator subgroup ‡ is the identity. It is clear that H and G cannot coincide, since G was assumed to have more than k independent generating operations.§

It is a well-known fact that there exist groups of class k whose conjugate operations are commutative for values of k ranging from 1 to 3.‖ We shall now show that no groups exist for $k>3$ in which conjugate operations are commutative. To establish this result it will be sufficient to consider the case $k=4$, since a group of class k must contain a subgroup of class $k-1$. It was proved above that a group of class k with more than k independent generating operations must contain a subgroup of class k which has exactly k independent generating operations. Consequently we assume that the group

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* Burnside, *loc. cit.*, p. 29; this follows also from the definition of *class*.

† Burnside, *loc. cit.*, p. 30.

‡ *Transactions of the American Mathematical Society*, Vol. 7 (1906), p. 61.

§ Note that the number of independent generating operations of a group of order p^n is an invariant of the group. *Vide* Miller, *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 21.

‖ For $k=1$ and $k=2$ we have the categories of abelian and metabelian groups, respectively. For $k=3$ we have the example exhibited by Fite, *loc. cit.*, p. 503.

$G_{(4)}$, whose conjugate operations are commutative, is of class 4 and has the four independent generating operations (S_1, S_2, S_3, S_4). The operations $S_{ij}, S_{ijk}, S_{ijkl}$ will be defined by the equations *

$$S_i^{-1}S_jS_i = S_jS_{ij}; \quad S_i^{-1}S_{jk}S_i = S_{jk}S_{ijk};$$
$$S_i^{-1}S_{jkl}S_i = S_{jkl}S_{ijkl}.$$

Burnside † has proved that the operation $S_{ab\ldots n}$ is changed into its inverse or remains unchanged according as an odd or an even permutation is effected on its subscripts. Now

$$S_{(12)(34)} = \overset{-1}{S_{34}}\overset{-1}{S_{12}}S_{34}S_{12}$$

is the inverse of $S_{(34)(12)}$. To show that $S_{(12)(34)} = \overset{-1}{S_{1234}}$ we proceed as follows:

$$\begin{aligned} S_{34}S_{(12)(34)} &= \overset{-1}{S_{12}}S_{34}S_{12} = \overset{-1}{S_1}\overset{-1}{S_2}S_1S_2 \cdot S_{34} \cdot \overset{-1}{S_2}\overset{-1}{S_1}S_2S_1 \\ &= \overset{-1}{S_1}\overset{-1}{S_2}S_1 \cdot S_{34}\overset{-1}{S_{234}} \cdot \overset{-1}{S_1}S_2S_1 = \overset{-1}{S_1}\overset{-1}{S_2} \cdot S_{34}\overset{-1}{S_{134}}\overset{-1}{S_{234}}S_{1234} \cdot S_2S_1 \\ &= \overset{-1}{S_1} \cdot S_{34}S_{234}\overset{-1}{S_{134}}\overset{-1}{S_{2134}}\overset{-1}{S_{234}}S_{1234} \cdot S_1 = \overset{-1}{S_1} \cdot S_{34}\overset{-1}{S_{134}}\overset{-1}{S_{2134}}S_{1234}S_1 \\ &= S_{34}S_{134}\overset{-1}{S_{134}}\overset{-1}{S_{2134}}S_{1234} = S_{34}\overset{2}{S_{1234}} = S_{34}\overset{-1}{S_{1234}}. \end{aligned}$$

Consequently ‡ $S_{(12)(34)} = \overset{-1}{S_{1234}}$. In the same manner we may show that $S_{(34)(12)} = \overset{-1}{S_{3412}}$. Since $S_{(12)(34)} = \overset{-1}{S_{(34)(12)}}$, we must have $\overset{-1}{S_{3412}} = S_{1234}$. But S_{3412} may be derived from S_{1234} by an even permutation on its subscripts; therefore $\overset{-1}{S_{1234}} = S_{1234}$, which is impossible, since S_{1234} is of order 3.§ From this contradiction we have the theorem: *There exists no finite group G of class greater than* 3 *in which conjugate operations are commutative.*

2. *Groups of Class* 3. In Art. 1 we proved that the group G of class k has exactly k independent generating operations or else contains a subgroup with this property. Consequently we shall investigate only those groups of

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* Burnside defines the symbol P_{ab} by the equation $P_b^{-1}\,P_a\,P_b = P_a\,P_{ab}$. It is evident that this notation differs only superficially from that given above.

† *Loc. cit.*, p. 31.

‡ In the reduction above use was made of the following: $S_{12} = S_2^{-1}\,S_1^{-1}\,S_2\,S_1$; two operations $S_{ij\ldots n}$ and $S_{ab\ldots l}$ are commutative if their subscripts have a letter in common; the order of S_{ijkl} is 3; the symbol S_{ijkl} is changed into itself or into its inverse according as an even or an odd permutation is effected on the letters in its subscript. The last three results are due to Burnside.

§ Burnside, *loc. cit.*, p. 31.

class k which have k independent generating operations. We are not interested in groups of class 1 or 2, which are contained in the categories of abelian and metabelian groups, respectively, and we have proved that G cannot be of class greater than 3. We assume, therefore, that G is generated by S_1, S_2, and S_3 of orders 3^{m_1}, 3^{m_2}, 3^{m_3}, respectively.* In the way of notation let the symbols c_{ij}, c_{kij} be defined by the equations $s_i^{-1}s_js_i = s_jc_{ij}$, $s_k^{-1}c_{ij}s_k = c_{ij}c_{kij}$.

Now from the results of Burnside † we have the following relation: (a) s_j is commutative with c_{ij} and c_{ki}; (b) the c's are relatively commutative; (c) $c^3_{ijk} = 1$; (d) $c_{ijk} = c_{kij} = c_{jik}^{-1}$ etc., that is, an even permutation on the symbols of the subscript of any c leaves it unchanged, while an odd permutation changes c into its inverse.

These relations evidently form a set of necessary conditions that G be of class 3. That they are also sufficient will be seen in the following paragraph. But we also obtain a set of sufficient conditions if we replace (c) by the weaker condition that c_{ijk} be invariant under the group $\{s_1, s_2, s_3\}$.

We assume that there exists a group $\Gamma = \{s_1, s_2, s_3\}$ which has the following properties: (a) s_1, s_2, s_3 are of orders 3^{m_1}, 3^{m_2}, 3^{m_3}, respectively; (b) s_i is commutative with every commutator of Γ whose subscript contains i; (c) the commutators of Γ are relatively commutative; (d) $c_{ijk} = c_{kij} = c_{jki} = c_{jik}^{-1} = c_{kji}^{-1} = c_{ikj}^{-1}$; (e) c_{ijk} is invariant in Γ. The notation will be made clearer by reference to the following equations:

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\begin{aligned}
s_1^{-1}s_2s_1 &= s_2c_{12} & \qquad s_1^{-1}c_{23}s_1 &= c_{23}k \\
s_2^{-1}s_3s_2 &= s_3c_{23} & \qquad s_2^{-1}c_{31}s_2 &= c_{31}k \\
s_3^{-1}s_1s_3 &= s_1c_{31} & \qquad s_3^{-1}c_{12}s_3 &= c_{12}k
\end{aligned}
$$

where $k = c_{123}$.

It is not difficult to see that any operation in Γ can be written in the form $s_1^{x_1} s_2^{x_2} s_3^{x_3} c_{12}^{y_3} c_{23}^{y_1} c_{31}^{y_2} k^{z}$. ‡

Let $S = s_1^{a_1} s_2^{a_2} s_3^{a_3} c_{12}^{\alpha_3} c_{23}^{\alpha_1} c_{31}^{\alpha_2} k^{\gamma}$ and $T = s_1^{b_1} s_2^{b_2} s_3^{b_3} c_{12}^{\beta_3} c_{23}^{\beta_1} c_{31}^{\beta_2} k^{\delta}$ be any two operations of Γ and let $S^{-1}TS = TC_{st}$ and $T^{-1}ST = SC_{ts}$. If we can show that $S^{-1}C_{st}S = C_{st}$ and $T^{-1}C_{st}T = C_{st}$, then it must follow that any

* Since Burnside proved that G is metabelian if its order is p^m where $p \neq 3$, and, further, that G is always the direct product of its Sylow subjects, we shall consider only the case where the order of G is a power of 3.

† *Loc. cit.*, pp. 29-31.

‡ Burnside, *loc. cit.*, p. 32.

two conjugate operations of Γ are commutative.* One may easily verify the equation

$$S^{-1}TS = Tc_{12}^{-a_2b_1+a_1b_2}\, c_{23}^{-a_3b_2+a_2b_3}\, c_{31}^{-a_1b_3+a_3b_1}k^{\lambda}$$

where $\lambda = -a_1b_1 - a_2b_2 - a_3b_3 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - a_1a_2b_3 - a_2a_3b_1 + a_3b_1b_2 + a_1a_3b_2 - a_2b_1b_3$.

Also, $S^{-1}C_{st}S = C_{st}k^{-a_1a_3b_2+a_1a_2b_3}k^{-a_1a_2b_3+a_2a_3b_1}k^{-a_2a_3b_1+a_1a_3b_2} = C_{st}$. Therefore S is commutative with C_{st}.

Moreover, $T^{-1}S_{st}T = C_{st}k^{-a_3b_1b_2+a_2b_1b_3}\, k^{-a_1b_2b_3+a_3b_1b_2}\, k^{-a_2b_1b_2+a_1b_2b_3} = C_{st}$. Accordingly T is commutative with C_{st}.

From this result that conjugate operations of Γ are commutative it is easy to show that the order of any operation of Γ is a power of 3. Accordingly, if conditions $(a), \cdots, (e)$ are fulfilled, the group $\{s_1, s_2, s_3\}$ is a group of order 3^m of class 3 whose conjugate operations are commutative. Thus the sufficiency of these conditions is demonstrated. From the necessary conditions obtained by Brunside for such groups it follows that the invariant operation k is of order 3.

3. *A Method of Constructing a Group of Class* 3. That groups of order 3^m and class 3 whose conjugate operations are commutative do not constitute a null-class was demonstrated by the example given by Fite.† We shall now develop a general method of constructing groups of this character which have three independent generating operations.

Let s_2 be a substitution of order 3^{m_2} on the letters $a_1, \cdots, a_{3a}$. Let c_{12} be any substitution of order $3^{m_{12}}$, $m_{12} \leqq m_2$, which is commutative with s_2. The group $\{s_2, c_{12}\}$ admits an automorphism in which s_2 corresponds to s_2c_{12} while c_{12} corresponds to itself. We write $\{s_2, c_{12}\}$ as a regular group and denote by s_1' a substitution of order $3^{m_{12}}$ in the holomorph of $\{s_2, c_{12}\}$ which effects this automorphism. Let s_1'' be any substitution of order $3^{m_1''}$ on a set of letters distinct from the letters of $\{s_2, c_{12}\}$. The substitution s_1'' is therefore commutative with s_1' and with every substitution of $\{s_2, c_{12}\}$. Now $s_1 = s_1'\, s_1''$ is a substitution of order 3^{m_1}, where m_1 is equal to the larger of the two exponents m_{12} and m_1''. One readily sees that the group $\{s_2, c_{12}, s_1\} = \{s_1, s_2\}$ is a metabelian group whose order is a power of 3.

We can find a substitution k of order 3 which is commutative with every operation of the metabelian group $\{s_1, s_2\}$. The group $\{s_1, s_2, k\}$ admits an automorphism in which $s_1 \sim s_1$; $k \sim k$; $c_{12} \sim c_{12}$; $s_2 \sim s_2k$. We write

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* Observe that $C_{st} = C_{st}^{-1}$; hence the two equations above involve the two additional equations $S^{-1}C_{ts}S = C_{ts}$ and $T^{-1}C_{ts}T = C_{ts}$.

† Fite, *loc. cit.*, p. 503.

$\{s_1, s_2, k\}$ as a regular group and denote by c_{31}' a substitution of order 3 in its holomorph which effects this automorphism. Let c_{31}'' be any substitution of order $3^{m_{31}''}$ on a set of letters distinct from the letters of $\{s_1, s_2, k\}$, where m_{31}'' does not exceed m_1. It is clear that c_{31}'' is commutative with c_{31}' and with every substitution of $\{s_1, s_2, k\}$. The order of $c_{31} = c_{31}' \, c_{31}''$ is $3^{m_{31}}$, where m_{31} is the larger of the two exponents m_{31}'' and 1. It is obvious that $\{s_1, s_2, k, c_{31}\} = \{s_1, s_2, c_{31}\}$ is a group whose order is a power of 3.

The group $\{s_1, s_2, c_{31}\}$ admits an automorphism which is defined by the following correspondences:

$$s_1 \sim s_1 k^{-1}; \; s_2 \sim s_2; \; c_{12} \sim c_{12}; \; c_{31} \sim c_{31}, \; k \sim k.$$

Let c_{23}' be a substitution of order 3 in the holomorph of the regular group $\{s_1, s_2, c_{31}\}$ which brings about this automorphism. Let c_{23}'' be a substitution of order $3^{m_{23}''}$ on a set of letters distinct from the letters of the regular group $\{s_1, s_2, c_{31}\}$, where $3^{m_{23}''}$ is not to exceed the order of s_2. Then $c_{23} = c_{23}' \, c_{23}''$, of order $3^{m_{23}}$, transforms the substitutions of $\{s_1, s_2, c_{31}\}$ according to c_{23}', since c_{23}'' is commutative with every substitution of $\{s_1, s_2, c_{31}\}$ as well as with c_{23}'. It is clear that $\{s_1, s_2, c_{31}\}$ and c_{23} together generate a group whose order is a power of 3.

One may easily verify the fact that the following correspondences define an antomorphism of $\{s_1, s_2, c_{31}, c_{23}\}$:

$$s_1 \sim s_1 c_{31}; \; s_2 \sim s_2 c_{23}^{-1}; \; c_{12} \sim c_{12} k; \; c_{31} \sim c_{31}; \; c_{23} \sim c_{23}; \; k \sim k.$$

[$s_1^{-1} s_2 s_1 = s_2 c_{12}$; $c_{31}^{-1} s_1^{-1} s_2 c_{23}^{-1} s_1 c_{31} = c_{31}^{-1} s_2 c_{12} c_{23}^{-1} k^{-1} c_{31} = s_2 k^{-1} c_{12} c_{23}^{-1} k^{-1}$ $= s_2 c_{12} c_{23}^{-1} k$, since $k_3 = 1$. This proves that the operation corresponding to $s_1^{-1} s_2 s_1$ is the same as the operation corresponding to $s_2 c_{12}$. It is not difficult to show that this is true for the remaining conjugate operations.]

Let s_3' be any substitution of lowest order in the holomorph of the regular group $\{s_1, s_2, c_{31}, c_{23}\}$ which transforms the substitutions of this group according to the automorphism above. The order of s_3', which we denote by $3^{m_3'}$, is evidently equal to the larger of the two numbers $3^{m_{31}}$ and $3^{m_{23}}$, where $3^{m_{31}}$ and $3^{m_{23}}$ are the orders of c_{31} and c_{23}, respectively. Let s_3'' be any substitution of any order $3^{m_3''}$ on a set of letters different from those of the regular group $\{s_1, s_2, c_{31}, c_{23}\}$. Then $s_3 = s_3' \, s_3''$ is of order 3^{m_3} where m_3 is the larger of the two exponents m_3' and m_3''. Now s_3 and $\{s_1, s_2, c_{31}, c_{23}\}$ generate a group of order 3^m which has the three independent generating operations s_1, s_2, s_3. From the mode of construction it is clear

that its operations satisfy the sufficient conditions which were developed in Art. 2. Consequently $G = \{s_1, s_2, s_3\}$ is a group of order 3^m and class 3 whose conjugate operations are commutative.

We shall now give two examples of such groups, each of order 3^7.

Example 1: In the notation of the preceding development we take $m_2 = 1$, $m_2 = 1$, $s_1'' = 1$. Then $\{s_1, s_2\}$ is a metabelian group of order 3^3 whose operations are all of order 3. Next, we take $c_{31}'' = 1$, and obtain the group $\{s_1, s_2, k, c_{31}\}$ of order 3^5, whose operations are all of order 3. Then we take $c_{23}'' = 1$, from which it follows that the group $\{s_1, s_2, k, c_{23}\}$ is of order 3^6 and contains only operations of order 3. Finally, we take $s_3'' = 1$ and obtain the group $\{s_1, s_2, s_3\}$ of order 3^7 whose operations are all of order 3. This is the group given by Fite.*

Example 2: Let s_2 and s_0 be two commutative operations of order 9 which have no powers in common except the identity. There exists an automorphism of $\{s_2, s_0\}$ in which s_0 corresponds to itself while s_2 corresponds to $s_2 s_0$. Let s_1 be an operation of order 9 in the holomorph of $\{s_2, s_0\}$ which effects this automorphism. The group $\{s_2, s_0, s_1\} \equiv \{s_2, s_1\}$ is a metabelian group of order 3^6 which admits an automorphism defined by the correspondences $s_1 \sim s_1 s_1^{-3}$, $s_2 \sim s_2 \cdot s_2^{-3}$, $s_0 \sim s_0 \cdot s^3_0$. The holomorph of $\{s_1 s_2\}$ contains an operation s_3 of order 3 which brings about this automorphism. It is clear that s_1, s_2, and s_3 together generate a group of order 3^7 in which the sufficient conditions of Art. 2 are satisfied. This group is of a more special type than the preceding example, in view of the fact that certain of the commutators are powers of generating operations. In the notation of the preceding example, we have the relations $c_{23} = s_2^3$, $c_{31} = s_1^{-3}$, $k = c_{12}^3$.

* *Loc. cit.*, p. 503.

Covariant Conditions for Multiple Roots of a Binary Form.

By L. T. Moore and J. I. Tracey.

If a binary form $(\alpha x)^n = 0$ has the roots $\alpha_1, \alpha_2, \cdots, \alpha_n$, any symmetric function of the differences of the roots and the differences between x and one or more of the roots is a covariant, provided each root enters the same number of times into the expression. By the proper use of this method invariants and covariants may be formed which will vanish on the hypothesis of any system of equalities between the roots. Cayley * has given a method of finding the conditions for the existence of any system of equalities among the roots of an equation, and has given the complete solution for the biquadratic and the quintic. For the general equation of order n the condition for a double root is well known, and the covariants which vanish for two double roots or for a triple root have been found.† This paper gives a method whereby a covariant may be formed which vanishes when an n-ic has a p-fold root, a p_1-fold root, a p_2-fold root, $\cdots$, a p_j-fold root simultaneously. It also enables one to determine the other systems of equality among the roots for which the same covariant will vanish.

1. *The condition for the existence of a p-fold root.* Let the order of the equation be m. The discriminant of the m-ic, which contains $\frac{1}{2}\, m \cdot \overline{m-1}$ factors of the type $(\alpha_1 - \alpha_2)^2$, may be represented by means of a diagram consisting of m points and having each of the m points connected by lines with the remaining $m-1$ points. Then each factor of the above type corresponds to one of the lines in the diagram.

Next, divide the m points into two sets of q and $m-q$ points each, where $2q \lesseqgtr m$. Connect each of the $m-q$ points with all the remaining points of that set, and each of the q points with the remaining $q-1$ points, but no point of either set is to be connected with any point of the other set. The product of terms corresponding to all the lines in the diagram will involve each of the $m-q$ roots to the degree $2(m-q-1)$ and each of the q roots to the degree $2(q-1)$. If this expression is multiplied by q factors of the type $(x-\alpha_i)^{2(m-2q)}$, where each α_i corresponds to one of the q set of roots, the product is a typical term of a summation which is a covariant of degree $2(m-q-1)$ and of order $2q(m-2q)$ which will be designated in the usual form $C_{2(m-q-1),\ 2q(m-2q)}$.

* *Collected Mathematical Papers*, Vol. 2, p. 465.

† Tracey, *American Journal of Mathematics*, Vol. 36, p. 31.

Obviously every term of this covariant will vanish if the m-ic has a triple root or root of greater multiplicity. It will not vanish for any number of double roots less than or equal to q, but will vanish if there are more than q double roots. Hence it cannot be made to vanish for any possible number of double roots provided $m = 2q$ or $m = 2q + 1$.

If the m points can be arranged into sets and connected in such a way that for any p points chosen at least two of them will be connected, and further, that some $p - 1$ of the m points may be chosen which are unconnected, then an invariant or covariant formed by the above method will vanish if the m-ic has a p-fold root but will not vanish if the multiplicity of the root is less than p. Divide the m points into $p - 1$ sets so that no two sets differ by more than one in the number of points which they contain; then connect each point of every set with the remaining points in that set but with no other point. The covariant or invariant which corresponds to this arrangement will vanish for a p-fold root or roots of greater multiplicity, but will not vanish for any combination of simultaneous multiple roots provided that each has a multiplicity of less than p. This statement is readily seen to be true, for if the multiplicity of the roots be $s_1, s_2, \cdots, s_g < p$ and $s_1 + s_2 + s_3 + \cdots + s_g \gtrless m$ these roots can be distributed among the $p - 1$ sets so that no set contains two roots which belong to the same s_i.

If m is a multiple of $p - 1$ there will be the same number of roots in each set and the resulting form is an invariant of degree $2(r - 1)$ where $r = m/(p - 1)$. If m is not a multiple of $p - 1$ the form will be a covariant since factors $(x - \alpha_i)$ must be introduced to make each root occur to the same degree. Let $(m + k)/(p - 1) = r$ where k is an integer between 0 and $p - 2$ inclusive, then in the diagram there will be $p - k - 1$ sets which contain r points and k sets which contain $r - 1$ points. The covariant obtained will be of degree $2(r - 1)$ and of order $2k(r - 1)$. The following table gives the forms which vanish when the m-ic has the indicated multiple root:—

Double root,		$I_{2(m-1)}$.
Triple root,	m even	I_{m-2}.
	m odd	$C_{(m-1),(m-1)}$.
Four-fold root,	$m = 3r$,	$I_{2(r-1)}$.
	$m = 3r - 1$,	$C_{2(r-1),2(r-1)}$.
	$m = 3r - 2$,	$C_{2(r-1),4(r-1)}$.
p-fold root,	$m = (p - 1)r - k$,	$C_{2(r-1),2k(r-1)}$.

There are $p-1$ possible cases according to the integral value of k less than $p-1$ which makes $(m+k)/(p-1)$ an integer r.

A different distribution of the roots among the $p-1$ sets will give rise to other covariants than those indicated in the table. All covariants, however, which are represented by such unequal distributions will vanish for some combination of simultaneous multiple roots even though none of the roots are of multiplicity equal to p.

2. *The condition that an n-ic have a p-fold and a p_1-fold root.* If $p_1 \geqq p$, the covariant which vanishes when the n-ic has a p-fold and a p_1-fold root may be represented in a similar manner by a diagram of n points. Divide these into p_1 sets so that p_1-p+1 sets contain a single point each, and distribute the remaining $n-p_1+p-1$ points among the remaining $p-1$ sets exactly as the m points were distributed among the $p-1$ sets in the preceding section. It is evident from this arrangement that p_1 unconnected points can be selected, after which, not more than $p-1$ points can then be selected which are unconnected. The corresponding covariant will vanish for a p_1-fold and a p-fold root; it will vanish for a p_1+1-fold root alone, but will not vanish for any number of simultaneous multiple roots of multiplicity less than p. It will vanish, however, if the n-ic has g multiple roots of multiplicity $s_1, s_2, \cdots, s_g$, respectively, provided each s_i satisfies the relation $p_1 > s_i > p$ and provided furthermore that $s_1+s_2+s_3+\cdots+s_g > (g-1)(p-1)+p_1$. This statement readily follows from the fact that of any s_i unconnected points chosen $s_i-(p-1)$ must be chosen from the p_1-p+1 set which contain single points.

Let $m=n-p_1+p-1$, then the degree of the covariant desired is seen to be the same, in terms of m, as that of the covariant which vanishes when an m-ic has a p-fold root, and the order is the order of that covariant increased by $2(p_1-p+1)(r-1)$ where r is the greatest number of roots in any set. Thus if $m/(p-1)$ is an integer r the required form is $C_{2(r-1),2(r-1)(n-m)}$; or in general, if $(m+k)/(p-1)$ is an integer r where k is some integer between 0 and $p-2$ inclusive then the covariant desired is a

$$C_{2(r-1),2(r-1)(n-m+k)}.$$

3. *The condition for a p-fold, a p_1-fold, and a p_2-fold root.* An extension of the methods of the two previous sections leads to the following results. Let $p_2 > p_1 > p$, the diagram consists of p_2-p_1 sets which contain single points, of p_1-p+1 sets which contain two points, and $p-1$ sets which contain the remaining m points of either r or $r-1$ points in each set. The

covariant represented by this diagram will vanish for a $p_2 + 1$-fold root alone; for a q_1-fold root and a q_2-fold root simultaneously provided $q_1 + q_2 > p_1 + p_2$; for a q-fold a q_1-fold and a q_2-fold root simultaneously provided $q + q_1 + q_2 > p + p_1 + p_2$; and for g simultaneous roots of multiplicity $s_1, s_2, \cdots, s_g$, provided each $s_i > p$, and provided also that $s_1 + s_2 + s_3 + \cdots + s_g > (g-2)(p-1) + p_1 + p_2$. It will not vanish for any number of simultaneous multiple roots of multiplicity less than p.

If, as before, we call $(m+k)/(p-1)$ an integer r the covariant which vanishes for a p-fold, a p_1-fold, and a p_2-fold root is a

$$(1) \qquad C_{2(r-1),\, 2[(r-1)(p_2-p_1+k)+2(r-2)(p_1-p+1)]}.$$

Finally we can say that the covariant which vanishes when an n-ic has $j+1$ multiple roots of multiplicity $p, p_1, \cdots p_j$ respectively, where each $p_{i+1} > p_i > p$, is a

$$(2) \qquad C_{2(r-1),\, 2[(p_j-p_{j-1}+k)(r-1)+2(p_{j-1}-p_{j-2})(r-2)+\ldots+(j-1)(p_2-p_1)(r-j+1)+j(p_1-p+1)(r-j)]}.$$

(1) and (2) may be written in the alternate forms:

$$(1') \qquad C_{2(r-1),\, 2[(r-1)(n-m+k)-2(p_1-p+1)]}.$$

and

$$(2') \qquad C_{2(r-1),\, 2[(r-1)(n-m+k)-2(p_{j-1}+2p_{j-2}+\ldots+(j-1)p_1-\frac{j(j-1)}{2}(p-1))]}$$

respectively.

where $m = n - (p_1 + p_2 + \cdots + p_j) + j(p-1)$, and k is the lowest integer which makes $(m+k)/(p-1)$ an integer r.

4. *Another method of constructing Covariants.* There is another method of forming a covariant which will vanish when an n-ic has a p_1-fold, a p_2-fold, $\cdots$, a p_j-fold root simultaneously, where $p_1 + p_2 + \cdots + p_j = m$ and which will not vanish for any other combination of multiple roots of multiplicity $q_1, q_2, \cdots, q_k$ respectively provided $q_1 + q_2 + \cdots + q_k < m$. A typical term of the summation which gives the covariant in terms of the roots of the n-ic may be represented by a diagram as follows:

Divide the n points into two sets, the first containing $m-1$ points and the second containing $n-m+1$ points. Connect each point of the first set with every point of the second set. Connect also every point of the second set with every other point of that set, but no two points of the first set are to be connected. Calling the points in the first set α_i and in the second set

β_i, there will be three types of factors in each term of the covariant, namely $(\alpha_i - \beta_j)$, $(\beta_i - \beta_k)^2$ and $(x - \alpha_i)^{n-2}$. There will be $m - 1$ factors of the latter type in each term, and as each root enters to the degree $2n - m - 1$ the covariant obtained by this process is a

$$C_{2n-m-1,(m-1)(n-2)}.$$

We observe that the factors of the type $(\alpha - \beta)$ only need to enter linearly for the existence of this covariant. The particular advantage of this method over the former is when the number of simultaneous multiple roots is large and the multiplicity of each relatively small. Thus for example, if this covariant is formed for the maximum number of double roots, say k, then it will vanish only for some combination of multiple roots whose combined multiplicity is $2k$. However, if the covariant which vanishes for k double roots is formed by the first method it will also vanish for a triple root.

On Certain Finitely Solvable Equations Between Arithmetical Functions.

By E. T. Bell.

1. *Introduction and Summary.* One of the fascinations of arithmetic is that almost any one of its general theorems, no matter how old, will give something new and unexpected when properly treated. The origin of the present note is a conjecture by Descartes, put forth 290 years ago, and proved in 1911 by Dubouis. The connection between this and what follows will be pointed out in § 4. For the moment we may remark that the number of equations between arithmetical functions for which we can prove either that they have no solution or only a finite number of solutions, and in the latter case state the solutions, is extremely small. In general it is difficult to say anything significant about such an equation; a classic instance is that of the equation defining the perfect numbers. If however an arithmetical problem containing at least one statement involving 'all' or 'any' can be solved in more than one way, a comparison of two distinct modes of solution will yield the complete theory of an equation of the type described. The functions to be considered are as follows:

$F(n), \equiv$ the number of odd classes of binary quadratic (Gauss) forms for the negative determinant $-n$; $F_1(n), \equiv$ the number of even classes; $E(n), \equiv F(n) - F_1(n)$. In these definitions all the usual conventions are to hold, as in H. J. S. Smith's *Report*, or as in the classnumber relations in Dickson's *History.*

$\epsilon(n), \equiv 1$ or 0 according as n is or is not the square of an integer > 0.

$N_r(n), \equiv$ the number of representations of n as a sum of r squares with integer roots $\gtreqless 0$, the order of the squares in a particular representation being relevant.

$Z_r(n), \equiv$ the similarly defined function with the restriction that the roots of the squares are $\gtrless 0$.

$\sigma(n), \equiv$ the sum of all the divisors of n.

$\tau(n), \equiv$ the sum of the odd divisors of n.

$\xi(n), \equiv$ the excess of the number of $4k+1$ $(k \geqq 0)$ divisors of n over the number of $4k+3$ divisors.

$(n, r), \equiv$ the coefficient of x^r in $(1+x)^n$.

We shall need the known theorems,

$$N_1(n) = 2\epsilon(n), \quad N_2(n) = 4\xi(n), \quad N_3(n) = 12E(n),$$
$$N_4(n) = [2 + (-1)^n]\tau(n).$$

Among others we shall prove the following. Unity is not counted as a prime.

I. *If* $m \equiv 3$ *mod* 8, *the only solutions of* $4F(m) = \sigma(m)$ *are* $m = 3, 11$.

II. *If* $m \equiv 1$ *mod* 4, *the only solutions of*

$$6F(m) = \sigma(m) + 3\xi(m) - \epsilon(m)$$

are $m = 1, 5, 9, 17, 29, 41$.

III. *The only odd squares* m^2 *such that*

$$6F(m^2) = \sigma(m^2) + 3\xi(m^2) - 1$$

are $m^2 = 1, 9$.

IV. *The only numbers* $m \equiv 1$, *mod* 4, *that are not squares, such that*

$$6F(m) = \sigma(m) + 3\xi(m)$$

are $m = 5, 17, 29, 41$. *Hence the only primes* $p \equiv 1$ *mod* 4 *such that* $6F(p) = p + 7$ *are* $p = 5, 17, 29, 41$.

V. *The only primes* $q \equiv 3$ *mod* 4 *such that* $4F(q) = q + 1$ *are* $q = 3, 11$.

VI. *The only* $m \equiv 3$ *mod* 4 *such that* $2F(2m) = \sigma(m)$ *are* $m = 3, 7$.

VII. *The only primes* q *of the form* $4k + 3$ *such that* $2F(2q) = q + 1$ *are* $q = 3, 7$.

VIII. *The only* $m \equiv 1$ *mod* 4 *such that* $2F(2m) = \sigma(m) + \xi(m)$ *is* $m = 1$.

IX. *If* p *is a prime of the form* $4k + 1$, $2F(2p) = p + 3$ *has no solution.*

X. *If* $m \equiv 1$ *mod* 4, $6F(m) = 3\sigma(m) + 3\xi(m) - \epsilon(m)$ *has no solution.*

Some of these are of course immediate corollaries of others. Nevertheless they are simple enough to merit separate statement. All follow from Descartes' conjecture which was proved by Dubouis, combined with the theorem

$$\text{XI.} \quad Z_r(n) = \sum_{j=0}^{r-1} (-1)^j \, (r, j) \, N_{r-j}(n),$$

which we shall prove.

2. *An algebraic lemma.* Let x be the umbra of the one-rowed matrix $(x_0, x_1, x_2, \cdots)$, and similarly for y and $(y_0, y_1, y_2, \cdots)$. Then, as always in the symbolic or umbral calculus, $x^n \equiv x_n$, and $x = y$ means only $x_n = y_n$ $(n = 0, 1, 2, \cdots)$. Each of $x = y + 1$ and $y = x - 1$ implies the other. That is,

$$x_n = (y+1)^n \equiv \sum_{j=0}^{n} (n,j) y_{n-j} \qquad (n = 0, 1, \cdots)$$

implies and is implied by

$$y_n = (x-1)^n \equiv \sum_{j=0}^{n} (-1)^j (n,j) x_{n-j} \qquad (n = 0, 1, \cdots).$$

If in particular $y_0 = 0$, we see that

$$\sum_{j=0}^{n-1} (n, n-j) y_{n-j} = 0 \qquad (n = 1, 2, \cdots, r)$$

implies

$$y_r = \sum_{j=0}^{r-1} (-1)^j (r,j) x_{r-j}.$$

Incidentally it may be noted that the last gives immediately the expansion of the determinant

$$\begin{vmatrix} x_r & (r, r-1) & (r, r-2) & \cdots & (r, 1) \\ x_{r-1} & (r-1, r-1) & (r-1, r-2) & \cdots & (r-1, 1) \\ x_{r-2} & 0 & (r-2, r-2) & \cdots & (r-2, 1) \\ x_{r-3} & 0 & 0 & \cdots & (r-3, 1) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_1 & 0 & 0 & \cdots & (1, 1) \end{vmatrix}$$

as the stated value for y_r.

3. *Proof of* § 1, XI. Let n denote an arbitrary constant integer > 0. Write N for the umbra of $(N_r(1), N_r(2), \cdots, N_r(n), \cdots)$, and similarly for Z and $Z_r(n)$ $(n = 1, 2, \cdots)$. Then if $|q| < 1$, and $S \equiv 2\sum_{n=1}^{\infty} q^{n^2}$, the coefficient of q^n in the expansion of S^r is $Z_r(n)$, and that of q^n in the expansion of $(1+S)^r$ is $N_r(n)$. Hence, from the identity.

$$(1+S)^r \equiv 1 + (r,1)S + (r,2)S^2 + \cdots + (r,r)S^r,$$

valid for all integers $r \geqq 0$, we infer, upon comparing coefficients of q^n, that $N_r(n) = (r,1)Z_1(n) + (r,2)Z_2(n) + \cdots + (r,r)Z_r(n)$. Now every integer is a sum of integer squares in at least one way. Hence $Z_0(n) = N_0(n) = 0$, and therefore we may write the last as

$$N_r(n) = \sum_{j=0}^{r} (r,j) Z_{r-j}(n),$$

which is valid for $r = 0, 1, 2, \cdots$. Hence $N = 1 + Z$; whence, $Z = N - 1$, and by § 2 we have § 1, XI.

4. *Descartes' conjecture.** In 1638 Descartes stated in letters to Mer-

* See Dickson, *History of the Theory of Numbers*, Vol. 2, p. 276; *ibid.*, p. 302, for the reference to Dubouis.

senne that he believed the only numbers n not a sum of 4 squares > 0 to be

$$1, 3, 5, 9, 11, 17, 29, 41, 4^h\lambda \qquad (\lambda = 2, 6, 14),\ h \geqq 0.$$

This conjecture was proved by Dubouis in 1911.

From § 1, XI, with $r = 4$, we get

$$Z_4(n) = N_4(n) - 4N_3(n) + 6N_2(n) - 4N_1(n),$$

the left of which vanishes only for the above values of n. Referring to § 1 for $N_j(n)$ $(j = 1, \cdots, 4)$, we see that

$$[2 + (-1)^n]\,\tau(n) + 3\xi(n) = 6E(n) + \epsilon(n)$$

is solvable only for the values of n stated by Descartes.

Incidentally we note that $4^h(8k+7)$ *is the sum of* 4 *squares with roots* $\gtrless 0$ *in* 8 *or* 24 *times* $\sigma(8k+7)$ *ways, according as* $h = 0$ *or* $h > 0$. For, the stated number is not a square or a sum of 2 or 3 squares, and the theorem follows from $Z_4(n) = N_4(n)$, $n \equiv 4^h(8k+7)$.

5. *Proofs of* § 1, *I-X.* From § 4 we see that the only odd integers $n > 0$ satisfying

$$\sigma(n) + 3\xi(n) = 6E(n) + \epsilon(n)$$

are $n = 1, 3, 5, 9, 11, 17, 29, 41$. We recall that $\xi(n)$ vanishes for $n = 2^h(4k+3)$; evidently $\epsilon(n) = 0$ for the same n. Again, $E(8k+7) = 0$, $3E(8k+3) = 2F(8k+3)$. Hence § 1, I follows; V is a corollary. Since $E(4k+1) = F(4k+1)$, § 1, II is proved, and III, IV are corollaries.

From the even values of n given as in § 4 by Descartes, we have the following:

XII. *If* $n = 2^{2h+k}\mu$, $h \geqq 0$, $k > 0$, *and if* $\mu \equiv 3 \bmod 4$, *the only solutions of*

$$2E(2^{2h+k}\mu) = \sigma(\mu)$$

are $h \geqq 0$, $k = 1$, $\mu = 3$ *and* $h \geqq 0$, $k = 1$, $= 7$.

XIII. *With h, k as in XII, and* $\mu \equiv 1 \bmod 4$, *the only solutions of*

$$3\sigma(\mu) + 3\xi(\mu) = 6E(2^{2h+k}\mu) + \epsilon(2^{2h+k}\mu)$$

are $h \geqq 0$, $k = 1$, $\mu = 1$.

XIV. *If* $p \equiv 1 \bmod 4$ *is prime,* $2E(2^{2h+k}p) = p + 3$ *has no solution.*

Since $E(4n) = E(n)$, it is sufficient to consider XII-XIV with $h = 0$. From XII we thus get § 1, VI, and VII is a corollary. Similarly XIII gives § 1, VIII, to which IX is a corollary.

In the equation in XIII take $k = 2$. Since $\epsilon(2^2\mu) = \epsilon(\mu)$, we get § 1, X.

On the Number of Representations of Integers by Certain Ternary Quadratic Forms.

By J. V. Uspensky.

In a recent paper * Professor L. E. Dickson has considered the possibility of representing integers by certain positive ternary quadratic forms. For the forms considered by Professor Dickson as well as for some other particular forms it is possible, however, to give the exact number of representations. The number of cases when the exact number of representations can be given seems to be limited, wherefore the discussion of such cases may present certain interest. In the following we consider two groups of closely related forms. The first group includes the following forms:

(I)	$x^2 + 2y^2 + 3z^2$	(V)	$x^2 + y^2 + 3z^2$
(II)	$x^2 + 2y^2 + 6z^2$	(VI)	$x^2 + 3y^2 + 3z^2$
(III)	$2x^2 + 3y^2 + 6z^2$	(VII)	$x^2 + y^2 + 6z^2$
(IV)	$x^2 + 3y^2 + 6z^2$	(VIII)	$x^2 + 6y^2 + 6z^2$,

whose coefficients do not contain other prime factors except 2 and 3. Similarly, the second group, consisting of 6 forms:

(IX)	$x^2 + y^2 + 5z^2$	(XII)	$x^2 + 5y^2 + 10z^2$
(X)	$x^2 + 5y^2 + 5z^2$	(XIII)	$x^2 + 2y^2 + 10z^2$
(XI)	$x^2 + 2y^2 + 5z^2$	(XIV)	$2x^2 + 5y^2 + 10z^2$

contains only forms whose coefficients involve prime factors 2 and 5.

1. *Forms* (I), (II), (III), (IV). All these forms are closely related to the form $x^2 + y^2 + z^2$. The number of representations of integers by this form was first determined by Gauss in the fifth section of Disquisitiones as an application of his general theory of ternary quadratic forms. In a more elementary way the same result may be derived from Kronecker's classnumber relations as shown by Kronecker himself. Finally, there exists a very simple and lucid deduction, based on the arithmetic of quaternions and due to Mr. Venkov.†

* L. E. Dickson, "Integers Represented by Positive Ternary Quadratic Forms," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 63-70.

† B. Venkov, "On the Arithmetic of Quaternions," *Bulletin of the Russian Academy of Sciences*, 1922.

Gauss' expression for the number of representations by the form $x^2+y^2+z^2$ involves two numerical functions $G(n)$ and $F(n)$ defined as follows: $G(n)$ is the number of all classes of positive binary quadratic forms corresponding to the determinant $-n$, this number being diminished by $\frac{1}{2}$ when n is a perfect square, and by $\frac{2}{3}$ when n is triple of a perfect square; similarly, $F(n)$ denotes the number of classes of all those forms, corresponding to the determinant $-n$ which can represent odd numbers with the exception that this number should be diminished by $\frac{1}{2}$ when n is a square of an *odd* number. It is also customary to set

$$G(0) = -(1/12), \qquad F(0) = 0.$$

This being so, we have in all cases

$$N(n = x^2 + y^2 + z^2) = 12[2F(n) - G(n)]. \tag{1}$$

Starting from this fundamental result it is not difficult to show that

$$N(n = x^2 + y^2 + 2z^2) = 4(2 + (-1)^n)[2F(2n) - G(2n)] \tag{2}$$

and

$$N(n = x^2 + 2y^2 + 2z^2) = 4(2 - (-1)^{(n-1)/2})[2F(n) - G(n)] \tag{3}$$

for an *odd* n, while

$$N(n = x^2 + 2y^2 + 2z^2) = 4(2 + (-1)^{n/2})[2F(n) - G(n)] \tag{4}$$

for an *even* n. Now, making use of the relation

$$N(3k = x^2 + 2y^2) + N(k/3 = x^2 + 2y^2) = 2N(k = x^2 + y^2) \tag{5}$$

holding true for every k * we find

$$2N(n = x^2 + 2y^2 + 6z^2) = N(3n = x^2 + 2y^2 + 18z^2) + N(n/3 = x^2 + 2y^2 + 2z^2).$$

Again,

$$N(3n = x^2 + 2y^2 + 18z^2) = \tfrac{1}{2}[N(3n = x^2 + 2y^2 + 2z^2) - N(n/3 = x^2 + 2y^2 + 2z^2)],$$

so that

$$N(n = x^2 + 2y^2 + 6z^2) = \tfrac{1}{4}N(3n = x^2 + 2y^2 + 2z^2) + \tfrac{3}{4}N(n/3 = x^2 + 2y^2 + 2z^2), \tag{6}$$

for every n. This way the number of representations by the form $x^2+2y^2+6z^2$

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* In the following to avoid separate consideration of different cases we suppose that all the numerical functions we have to consider vanish when their argument is a fractionary number.

may be determined by making use of (4), but it is hardly necessary to develop the final formulas.

For any number n we obviously have

$$N(2n = x^2 + 2y^2 + 6z^2) = N(n = x^2 + 2y^2 + 3z^2),$$

whence and from (6) it follows that

$$(7) \quad N(n = x^2 + 2y^2 + 3z^2) \\ = \tfrac{1}{4}N(6n = x^2 + 2y^2 + 2z^2) + \tfrac{3}{4}N(2n/3 = x^2 + 2y^2 + 2z^2),$$

where the right hand member may be expressed by means of (4).

The same relation (5) leads easily to the following equation:

$$N(3n = x^2 + 2y^2 + 6z^2) + N(n = 2x^2 + 3y^2 + 6z^2) \\ = 2N(n = x^2 + 2y^2 + 2z^2)$$

which, combined with (6), gives

$$N(n = 2x^2 + 3y^2 + 6z^2) = -\tfrac{1}{4}N(9n = x^2 + 2y^2 + 2z^2) \\ + (5/4)N(n = x^2 + 2y^2 + 2z^2).$$

Now, taking into account (4) as well as the general formulas

$$G(9m) = [4 + (m/3)]\, G(m) - 3G(m/9)$$
$$F(9m) = [4 + (m/3)]\, F(m) - 3F(m/9),$$

we finally find:

$$(8) \quad N(n = 2x^2 + 3y^2 + 6z^2) = \{[1 - (n/3)]/4\}\, N(n = x^2 + 2y^2 + 2z^2),$$

provided, n is not divisible by 3.

Finally, to determine the number of representations by the form $x^2 + 3y^2 + 6z^2$ we notice that

$$N(2n = 2x^2 + 3y^2 + 6z^2) = N(n = x^2 + 3y^2 + 6z^2)$$

and therefore, supposing n non-divisible by 3,

$$(9) \quad N(n = x^2 + 3y^2 + 6z^2) \\ = \{[1 + (n/3)]/4\}\, N(2n = x^2 + 2y^2 + 2z^2).$$

2. *Forms* (V), (VI), (VII), (VIII). These four forms are again intimately related. With respect to forms

$$x^2 + y^2 + 3z^2 \text{ and } x^2 + 3y^2 + 3z^2$$

we can confine ourselves to find the number of representations of integers which are not divisible by 3. To this end we resort to certain classnumber

relations established by elementary means in our paper "On Gierster's class-number relations."

Denoting by n any integer, we have in case of an odd n

$$\sum \mathrm{F}(3n - 9h^2) = (1/2)\phi(3n) - \phi(n) + (3/2)\psi(n/3)$$
$$\sum G(3n - 9h^2) = \phi(3n) - (8/3)\phi(n) + 3\psi(n/3)$$

while for an *even* n

$$\sum F(3n - 9h^2) = X(n) - 3\phi(n/12) + 3\psi(n/12)$$
$$\sum G(3n - 9h^2) = 2\phi(n/2) - 6\phi(n/12 + \psi(n/12)$$
$$\sum G(3n - 9h^2) = 2\phi(n/2) - 6\phi(n/12) + \psi(n/12).$$

Here, as usual, $\phi(n)$ represents the sum of all divisors of n, while $X(n)$ and $\psi(n)$ are defined by

$$X(n) = \sum d, \quad n = d\delta, \quad \delta \text{ odd}$$
$$\psi(n) = \sum (d' - d), \quad n = dd\,, \quad d < d.$$

At the same time

$$\sum F[(n/3) - h^2] = (1/2)\phi(n/3) + (1/2)\psi(n/3)$$
$$\sum G[(n/3) - h^2] = (1/3)\phi(n/3) + \psi(n/3)$$

for an odd n, and

$$\sum F[(n/3) - h^2] = X(n/3) - \phi(n/12) + \psi(n/12)$$
$$\sum G[(n/3) - h^2] = 2\phi(n/2) - 2\phi(n/12) + 2\psi(n/12)$$

for an even n. Putting

$$\omega(n) = 3G(3n) - 2F(3n) - 3[3G(n/3) - 2F(n/3)],$$

we derive from the preceding equations

$$2\sum \omega(n - 3h^2) = (-1)^n\, 4 \sum (-1)^d d,$$

the righthand member sum being extended over *all* the divisors of n, which are not divisible by 3.

On the other hand, we have

$$N(n = x^2 + y^2 + 3z^2 + 3t^2) = (-1)^n\, 4 \sum (-1)^d d,$$

so that, putting for brevity

$$N(n = x^2 + y^2 + 3z^2) = X(n),$$

we have for every n

$$2\sum \omega(n - 3h^2) = \sum X(n - 3h^2),$$

and this necessitates

$$X(n) = 2\omega(n),$$

that is,

$$N(n = x^2 + y^2 + 3z^2) = 2\omega(n).$$

In case n is not divisible by 3 this reduces to

$$(10) \qquad N(n = x^2 + y^2 + 3z^2) = 2[3G(3n) - 2F(3n)].$$

Now, if we substitute $3n$ instead of n and suppose $n \equiv 1 \pmod 3$, we get readily

$$(11) \qquad N(n = x^2 + 3y^2 + 3z^2) = 4[3G(n) - 2F(n)],$$

while for $n \equiv 2 \pmod 3$ we obviously have

$$N(n = x^2 + 3y^2 + 3z^2) = 0.$$

To find the number of representations by the remaining forms $x^2 + y^2 + 6z^2$ and $x^2 + 6y^2 + 6z^2$ we make use of the general identity (VIII) of our above mentioned paper. We assume in it

$$F(x, y, z) = 0 \text{ for an even } x$$
$$F(x, y, z) = (-1)^{(x-1/2)} \text{ for an odd } x,$$

and after a simple discussion we arrive at the following useful relations:

$$(12) \qquad \sum_{(a)} (-1)^x = \sum_{(b)} (-1)^x - [(-1)/2]^{n/2} \{N[3n = h^2 + 2k^2 + 2l^2] - N[(n/3) = h^2 + 2k^2 + 2l^2]\}$$

$$(13) \qquad \sum_{(a)} (-1)^x = \sum_{(b)} (-1)^x + [(-1)/2]^{(n-1)/2} \{N[3n = h^2 + 2k^2 + 2l^2] - N[(n/3) = h^2 + 2k^2 + 2l^2]\}$$

holding true for even and odd n respectively, the extent of summations being as follows:

$$\text{(a)} \quad n = 3x^2 + y^2 + z^2; \qquad \text{(b)} \quad (n/3) = x^2 + y^2 + z^2.$$

We first suppose n non divisible by 3; denoting by R and S respectively the number of solutions of the equations

$$n = 12x^2 + y^2 + z^2$$
$$n = 3x^2 + y^2 + z^2, \quad x \text{ odd},$$

we have,

$$R - S = -[(-1)/2]^{n/2} N(3n = h^2 + 2k^2 + 2l^2) \text{ for an even } n$$
$$R - S = [(-1)/2]^{(n-1)/2} N(3n = h^2 + 2k^2 + 2l^2) \text{ for an odd } n.$$

Now the righthand member can be found from (3) and (4), while the sum $R+S$ is given by (10). We can, therefore, find R and S separately and the final expression for R is

$$R=2(-1)^{n/2}[3G(3n)-4F(3n)] \text{ for an even } n$$
$$R=2(-1)^{(n-1)/2}[3G(3n)-4F(3n)] \text{ for an odd } n.$$

On the other hand, it is obvious that

$$R(2n)=N(n=x^2+y^2+6z^2)$$

and, using the preceding expression for R,

$$N(n=x^2+y^2+6z^2)=2(-1)^n[3G(6n)-4F(6n)] \tag{14}$$

for any number non-divisible by 3.

In a similar way, denoting by P and Q respectively the number of solutions of the equations

$$n=4x^2+3y^2+3z^2$$
$$n=x^2+3y^2+3z^2,\ x \text{ odd},$$

and replacing n by $3n$ [$n\equiv 1 \pmod{31}$ in (12) and (13)], we find

$$Q=(-1)^{(n-1)/2}4[4F(n)-3G(n)]$$

for an odd n and

$$P=(-1)^{n/2}4[3G(n)-4F(n)]$$

for an even n. But obviously

$$Q=N(n=x^2+6y^2+6z^2),$$

if n is odd and

$$P=N(n=x^2+6y^2+6z^2)$$

if n is even. That is, finally,

$$N(n=x^2+6y^2+6z^2)=(-1)^{(n+1)/2}4[3G(n)-4F(n)] \tag{15}$$
$$N(n=x^2+6y^2+6z^2)=(-1)^{n/2}4[3G(n)-4F(n)] \tag{16}$$

for an *odd* and *even* n respectively.

At the same time it is clear from the preceding analysis that the general expression for the number of representations by two forms $2x^2+2y^2+3z^2$ and $2x^2+3y^2+3z^2$ can be found, but it does not seem worth while to write down the final expressions.

3. *Forms* (*IX*) *and* (*X*). Both these forms are closely related to the form $x^2+y^2+z^2$. We start from the almost obvious relation

(17) $\quad N(5n = x^2 + y^2) + N(n/5 = x^2 + y^2) = 2N(n = x^2 + y^2)$

and, replacing n by $n - 5z^2$, summing up all the equations thus obtained by making z assume all integral values so that $n - 5z^2$ remains $\geqq 0$, we get

$$2N(n = x^2 + y^2 + 5z^2) = N(5n = x^2 + y^2 + 25z^2) + N(n/5 = x^2 + y^2 + z^2),$$

whence it follows,

(18) $$N(n = x^2 + y^2 + 5z^2) = (1/6)N(5n = x^2 + y^2 + z^2) + (5/6)N(n/5 = x^2 + y^2 + z^2),$$

which gives the number of representations by (IX).

Suppose now $n \equiv \pm 1 \pmod 5$. From the same relation (17), denoting by Q the number of solutions of the equation

$$n = x^2 + y^2 + z^2,$$

where

$$x^2 \equiv \pm 1 \pmod 5,$$

signs corresponding to those in $n \equiv \pm 1 \pmod 5$, we derive

$$Q + N(n = x^2 + 25y^2 + 25z^2) = 2N(n = x^2 + 5y^2 + 5z^2).$$

But it is easy to see that

$$Q + N(n = x^2 + 25y^2 + 25z^2) = (2/3)N(n = x^2 + y^2 + z^2),$$

whence it follows that

$$N(n = x^2 + 5y^2 + 5z^2) = (1/3)N(n = x^2 + y^2 + z^2)$$

for every $n \equiv \pm 1 \pmod 5$.

4. *Forms* (XI), (XII), (XIII), (XIV). We are going to determine first the number of solutions of the equation

$$8n = i^2 + 2j^2 + 5k^2$$

in odd numbers i, j, k. For this purpose we take for our starting point the equations (20) and (21) established in our paper "On Gierster's classnumber relations" already quoted. These equations may be written as follows:

$$\sum F(5n - 25h^2) = (1/4)N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) + (1/16)\,N(5n = x^2 + y^2 + z^2 + 25t^2) + (5/2)\psi(n/5)$$

for an *odd* n and

$$\sum F(5n - 25h^2) = (1/4)N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) + (1/16)N(5n = x^2 + y^2 + z^2 + 25t^2) + (5/2)\phi(n/10) + 5\psi(n/20)$$

for an *even* n. These relations, being combined with the known expressions for the sum

$$\sum F[n/5 - h^2],$$

give

$$\begin{aligned}\sum \{F(5n - 25h^2) - 5F[(n - 5h^2)/5]\} \\ = (1/4)N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) \\ - (5/16)N(n/5 = x^2 + y^2 + z^2 + t^2) \\ + (1/16)N(5n = x^2 + y^2 + z^2 + 25t^2),\end{aligned}$$

and this holds true whether n be odd or even.

On the other hand, we have found in the same paper (see equation (36)):

$$\begin{aligned}\sum (-1)^z = (5/4)N[n/5 = x^2 + y^2 + z^2] \\ - (1/4)N(5n = x^2 + y^2 + z^2),\end{aligned} \tag{20}$$

sum being extended over all solutions of the equation

$$8n = x^2 + 2y^2 + 5z^2,$$

whence we find easily

$$\begin{aligned}\sum Q(n - 5h^2) = N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) \\ - (5/4)N(n/5 = x^2 + y^2 + z^2 + t^2) + (1/4)N(5n = x^2 + y^2 + z^2 + 25t^2).\end{aligned}$$

Here $Q(n)$ stands for the number of solutions of the equation

$$8n = i^2 + 2j^2 + 5k^2$$

with *odd* i, j, k. Comparing this result with one previously found, we have

$$\sum Q(n - 5h^2) = 4 \sum \{F(5n - 25h^2) - 5F[(n - 5h^2)/5]\}$$

for every n, and hence we conclude

$$Q(n) = 4[F(5n) - 5F(n/5)]. \tag{21}$$

The value of $Q(n)$ being thus determined, we find from the same equation (20)

$$\begin{aligned}N(2n = x^2 + 2y^2 + 5z^2) = 3G(5n) - 2F(5n) \\ + 10F(n/5) - 15G(n/5),\end{aligned} \tag{22}$$

and this holds true for every n. Now, supposing first n non divisible by 5, we can write down the preceding equation as follows

$$N(n = x^2 + 2y^2 + 5z^2) = 3G(10n) - 4F(10n)$$

for every *even* number n non divisible by 5.

To find the number of representations of an *odd* number n we observe that

$$N(n = x^2 + 2y^2 + 5z^2) = N(4n = x^2 + 2y^2 + 5z^2)$$

The righthand member here is determined by the preceding equation, and, some obvious simplifications made, we get

$$(23) \qquad N(n = x^2 + 2y^2 + 5z^2) = (-1)^n\,[3G(10n) - 4F(10n)],$$

provided n is not divisible by 5.

Next, we replace n by $5n$ in (22) which gives

$$N(2n = x^2 + 5y^2 + 10z^2) = 3G(25n) - 2F(25n) + 10F(n) - 15G(n).$$

Here the righthand member can be simplified using the general relations

$$F(25n) = [6 - (n/5)]\,F(n) - 5F(n/25)$$
$$G(25n) = [6 - (n/5)]\,G(n) - 5G(n/25).$$

Confining ourselves to the most interesting case $2n \equiv \pm 1 \pmod 5$, we find

$$N(2n = x^2 + 5y^2 + 10z^2) = 2[3G(n) - 2F(n)],$$

or, which is the same,

$$N(n = x^2 + 5y^2 + 10z^2) = 2[3G(2n) - 4F(2n)]$$

for every *even* $n \equiv \pm 1 \pmod 5$. For an *odd* n we evidently have

$$N(n = x^2 + 5y^2 + 10z^2) = N(4n = x^2 + 5y^2 + 10z^2),$$

whence and from the preceding we derive the following general expression

$$N(n = x^2 + 5y^2 + 10z^2) = (-1)^n\,2[3G(2n) - 4F({}^2n)]$$

provided $n \equiv \pm 1 \pmod 5$.

In the same paper already quoted we established the following equations holding true for $n \equiv \pm 2 \pmod 5$:

$$N(2n = x^2 + 5y^2 + 10z^2) + 2(-1)^{(n-1)/2}\,N(n = 2x^2 + 5y^2 + 10z^2)$$
$$= (1/2)N(n = x^2 + y^2 + z^2)$$
$$N(2n = x^2 + 5y^2 + 10z^2) - 2(-1)^{n/2}\,N(n = 2x^2 + 5y^2 + 10z^2)$$
$$= (1/2)N(n = x^2 + y^2 + z^2)$$

for an *odd* and *even* n respectively. These equations may serve to obtain the number of representations by the form $2x^2 + 5y^2 + 10z^2$. We find

$$(25) \qquad N(n = 2x^2 + 5y^2 + 10z^2) = (-1)^{n/2}\,2[3G(n) - 4F(n)]$$
$$(26) \qquad N(n = 2x^2 + 5y^2 + 10z^2) = (-1)^{(n+1)/2}\,2[3G(n) - 4F(n)]$$

for an even and odd $n \equiv \pm 2 \pmod{5}$ respectively.

It remains to consider the form $x^2 + 2y^2 + 10z^2$. Supposing *n even,* we have the obvious relation

$$N(n = x^2 + 2y^2 + 10z^2) = N(n/2 = x^2 + 2y^2 + 5z^2),$$

whence and from (23) it follows,

$$(27) \qquad N(n = x^2 + 2y^2 + 10z^2) = (-1)^{n/2}\,[3G(5n) - 4F(5n)],$$

provided n is not divisible by 5.

To find the number of representations by the same form in case of an *odd* n we make use of the relation

$$\Sigma\,(-1)^z = -\,(1/4)\,\Sigma\,(-1)^{i+j},$$

where sums are extended over solutions of the equations

$$2n = x^2 + 2y^2 + 5z^2$$
$$5n = i^2 + j^2 + k^2$$

respectively and n denotes any number non-divisible by 5. This relation follows immediately from one established in § 11 of our paper quoted above. Let us denote by P and Q respectively the number of solutions of the equation

$$2n = x^2 + 2y^2 + 5z^2$$

with *even* and *odd* z. The preceding relation gives for $n \equiv 1 \pmod{4}$

$$P - Q = 2F(5n) - G(5n),$$

while

$$P + Q = 3G(5n) - 2F(5n),$$

whence

$$P = N(n = x^2 + 2y^2 + 10z^2) = G(5n).$$

For $n \equiv 3 \pmod{4}$ it follows from the same equation,

$$P - Q = 3G(5n) - 6F(5n)$$

and

$$P + Q = 3G(5n) - 2F(5n),$$

whence

$$P = N(n = x^2 + 2y^2 + 10z^2) = 3G(5n) - 4F(5n).$$

Both expressions thus obtained may be summarized in a single equation:

$$(28) \qquad N(n = x^2 + 2y^2 + 10z^2) = (-1)^{(n+1)/2}\,[3G(5n) - 4F(5n)],$$

provided n is an *odd* number non-divisible by 5.

Representation of Integers in the Form. $x^2 + 2y^2 + 3z^2 + 6w^2$.

By L. W. Griffiths.

1. *Introduction.* Jacobi * first proved that every positive integer N is represented in the form

$$(1) \qquad x^2 + 2y^2 + 3z^2 + 6w^2.$$

Liouville † gave the number of representations when N is even, a power of 3, or a multiple of 7, as a linear function of the sum of the divisors prime to 6 of N; he gave upper and lower limits only for other odd N. The problem for N odd is recognized now as an extremely difficult and important one. In this paper the results of Liouville for N odd are proved, and better limits than his are found. Secondly, although the methods of this paper seem to be ineffective in determining the number of representations when N is an odd prime different from 3, the problem for N odd and composite is reduced to that for N odd and prime.

In particular, the essential notations and results of this paper are as follows. Write $N = 3^b M$ where $b \geqq 0$ and M is prime to 6; define $f(M)$ as the sum of the distinct positive divisors of M, and $g(N)$ and $\beta(N)$ by $g(N) = \beta(N) f(M) = (3^{b+1} - 2) f(M)$. It is easily shown that the number of representations of N by (1) is even, in notation $2T(N)$. Then $T(N) \equiv g(N) \pmod 2$, and $T(N)$ satisfies Liouville's inequality $g(N)/3 \leqq T(N) \leqq g(N)$. Also $T(N) \leqq g(N) - 2t(N)$, where $t(N) = E(N) + k(N)$, $E(N)$ is the greatest integer in $(2N/3)^{1/2}$, and $k(N)$ is 0 or 1 according as $E(N)$ is even or odd. Hence a better inequality than Liouville's is $g(N)/3 \leqq T(N) \leqq g(N) - 2t(N)$. The problem is reduced to that for P^n, where P is an odd prime and n is a positive integer, since $D(N)$ defined to be $2T(N) - g(N)$ is a factorable function, that is $D(PQ) = D(P)D(Q)$ when P and Q are relatively prime. The problem is reduced to that of odd primes by $T(P^n) = T(P^{n-2}) + K(P)$, where $K(P)$ is a polynomial in $f(P)$ and $T(P)$. By this reduction of the problem to that for P^n, a third inequality, better than the second, is obtained for N composite but not a power of a prime. By the same reduction, $2T(N) = g(N)$ if N is divisible by 7, while for every $A \geqq 0$ there are infinitely many integers N such that $| 2T(N) - g(N) | > A$.

2. *Number of representations for N even.* In an earlier paper ‡ we

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* Dickson, *History of the Theory of Numbers*, Vol. 3, 229.

† *Journal de Mathématique*, Vol. 9, 2 série (1864).

‡ *American Journal of Mathematics*, Vol. 50 (1928), 303-314.

obtained the preceding results, of Jacobi and of Liouville for N even, in a discussion of generalized quaternion algebras. There the following notations and relations were established. The integral quantities of the algebra with parameters $(-2, -3)$ are the quantities $q = \kappa + \lambda i + \mu H + \nu L$ in which $\kappa, \lambda, \mu, \nu$ are integers; the conjugate of q is $\bar{q} = \kappa + \mu - \lambda i - \mu H - \nu L$ and the norm $q\bar{q}$ is the integer

$$(2) \qquad \kappa^2 + \kappa\mu + \mu^2 + 2\lambda^2 + 2\lambda\nu + 2\nu^2,$$

that is the value of (1) with

$$(3) \qquad x = \kappa + \mu/2, \qquad y = \lambda + \nu/2, \qquad z = \mu/2, \qquad w = \nu/2.$$

Every integer $N \geqq 0$ is represented by (1) since there is in the algebra a quantity q having μ and ν even and norm N. From $4 = x^2 + 2y^2 + 3z^2 + 6w^2$ and (3), the units are ± 1, $\pm H$, $\pm (1-H)$. Since the norm of a product is the product of the norms, all the quantities of norm N are associated in sets of six each, the quantities in any one set being the products qu where q is any specific one in the set and u is in turn each of the six units. q is proper (definition) if the greatest common divisor of $\kappa, \lambda, \mu, \nu$ is 1. Hence if q is a proper quantity having norm $N = P_1 P_2 \ldots P_h$ in which the P's are primes arbitrarily arranged, then there are quantities $p_1, p_2, \cdots, p_h$ of norms $P_1, P_2, \cdots, P_h$ respectively such that $q = p_1 p_2 \cdots p_h$; *and* each of the p's is successively unique up to a unit right factor if not more than one of the P's is 3. If the norm of q is not divisible by 3 then q is proper if and only if no factor p in the product is a right associate of the conjugate of its predecessor. Write $N = 2^a 3^b M$ ($a \geqq 0$, $b \geqq 0$, M prime to 6). Then the number of distinct quantities of norm N is $6g(N)$, that is the number of representations of N in (2) is $6g(N)$ and the number of sets of right associated quantities of norm N is $g(N)$. According as $a > 1$ or $a = 1$ six or two of the six right associates in each set have μ and ν even. Hence, as stated by Liouville, *the number of representations of N in* (1) *is* $6g(N)$ *if* $a > 1$, $2g(N)$ *if* $a = 1$.

3. *Number of Representations for N odd reduced to the case of N odd and prime.* Since $x^2 = (-x)^2$ this number of representations is even, in notation $2T(N)$. It is necessary to enumerate, for $N = 2^a 3^b M$ as above but with $a = 0$, those of the $6g(N)$ representations which have μ and ν even. Now $N \equiv 1 \pmod 2$ implies that κ and μ are not both even. Hence $\kappa, \lambda, \mu, \nu$ reduced modulo 2 determine one of the following quadruples

$$
(4) \qquad \begin{array}{cccc} 1\,0\,0\,0 & & 0\,0\,1\,0 & & 1\,0\,1\,0 \\ 1\,1\,0\,0 & & 0\,0\,1\,1 & & 1\,1\,1\,1 \\ 1\,1\,0\,1 & & 0\,1\,1\,0 & & 1\,0\,1\,1 \\ 1\,0\,0\,1 & & 0\,1\,1\,1 & & 1\,1\,1\,0. \end{array}
$$

Table (4) is basic in the sequel. It is easily verified that if q is in the first column then qH and $q(1-H)$ are in the second and third respectively of the same row; hence each of the $g(N)$ sets in § 2 corresponds to a row of the table. If q is in the first column Hq is in the second column of the row determined by the permutation $(1)(324)$, and $(1-H)q$ in the third column of the row determined by $(1)(342)$. Define $2R(N)$ and $2S(N)$ as the numbers of representations of the first column, first and second rows respectively. Then

$$
(5) \qquad T = R + S, \qquad g = R + 3S, \qquad D = R - S.
$$

The last equation defines $D(N)$; the first holds since only the first and second rows of the table correspond to μ and ν both even. The second holds since (2) is symmetrical in κ and μ, for then a representation in the second row determines one in the third and this in turn one in the fourth.

THEOREM 1. *If* $N = \prod_1^n M_h$, *where* n *is a positive integer and every two of the* M_h *are relatively prime, then*

$$
2T(N) = \prod_1^n g(M_h) + \prod_1^n [2T(M_h) - g(M_h)].
$$

Case $n = 2$. The products pq are the distinct quantities of norm M_1M_2 if p runs through the distinct (proper and not proper) quantities of norm M_1 and q similarly for M_2. To determine how many of these products pq have μ and ν even, that is are in the first column, first or second row, of (4), it is sufficient to take p and q each in the first column of (4). Then the following table gives the row of pq in (4), the row of p in (4) being read off down the left and that of q across the top:

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Now in the product pq, p may be chosen in the first column of (4) in $R(M_1)$ ways from the first row (written R_1) and in S_1 ways from each of the other three rows; and similarly for q. Hence by the diagonal of the table

(6) $R_{12} = R(M_1M_2) = R_1R_2 + S_1S_2 + S_1S_2 + S_1S_2 = R_1R_2 + 3S_1S_2,$

and similarly

(7) $S_{12} = S(M_1M_2) = R_1S_2 + S_1R_2 + S_1S_2 + S_1S_2 = R_1S_2 + S_1R_2 + 2S_1S_2.$

But by (5), (6) and (7)

$$D_{12} = R_{12} - S_{12} = D_1D_2, \quad 2T_{12} = g_1g_2 + D_1D_2.$$

Case $n > 2$. By the familiar properties of $f(N)$, since the M's are relatively prime, $g(M_2 \cdots M_n) = g_2 \cdots g_n$. Furthermore, as for D_{12} above, $D(M_2 \cdots M_n) = D_2 \cdots D_n$.

Hence $2T(M_1 \cdots M_n) = g_1 \cdots g_n + D_1 \cdots D_n$.

THEOREM 2. $2T(3^b) = \beta(3^b) + (-1)^b$,

$R(3^b) = [\beta(3^b) + 3(-1)^b]/4, \quad S(3^b) = [\beta(3^b) - (-1)^b]/4.$

By the reference in § 2, the distinct quantities of norm 3^b are known, and the non-proper among them are segregated and counted. Then (6) and (7) give $R(3^b) = R(3^{b-2}) + (3^b + 3^{b-1})/2, \qquad S(3^b) = S(3^{b-2}) + 2 \cdot 3^{b-1},$ from which the theorem follows.

Theorem 2 and theorem 1 with $M_1 = 3^b$ give Liouville's result,

$$2T(N) = g(N) + (-1)^b[2T(M) - f(M)], \quad a = 0, \quad b > 0.$$

THEOREM 3. *If* $n \geqq 2$ *and* P *is a prime* $\neq 2, 3$ *then:* $T(P^n) = T(P^{n-2}) + K(P)$, *where* $K(P)$ *is a polynomial in* $f(P)$ *and* $T(P)$; $T(P^n)$ *is a polynomial in* $f(P)$ *and* $T(P)$.

$2T(P^n)$ is less than the value from theorem 1 of $2T(M_1 \cdots M_n)$ with $M_1 = \cdots = M_n = P$. Define $2T_n'$ as the latter; and $2c_n$, $2d_n$, $2e_n$ as the numbers respectively of non-proper, proper, and distinct non-proper quantities enumerated by $2T_n'$. Then

$$\begin{aligned} &T'_n = c_n + d_n, \\ (8) \qquad &T(P^n) = d_n + e_n = T_n' - c_n + e_n, \\ &d_0 = 1, \; d_1 = T(P), \quad c_0 = c_1 = e_0 = e_1 = 0, \\ &\qquad\qquad c_2 = f, \; e_2 = 1, \quad d_2 = T(P^2) - 1. \end{aligned}$$

A quantity which is not proper is of the form $p\bar{p}q$, where the norm of q is P^{n-2} (this includes quantities factored, when enumerated in T_n', as $qp\bar{p}r$ having the norm of qr equal to P^{n-2}). In c_n there are fT'_{n-2} products $p\bar{p}q$ of norm P^n in which the first and second factors certainly make the product non-proper; for $h = 1, 2, \cdots n-2$ there are $d_h(f-1)T'_{n-2-h}$ non-proper products of norm P^n in which $p_1 \cdots p_h$ is proper. Hence

$$c_n = fT'_{n-2} + (f-1)\sum_h d_h T'_{n-2-h}, \quad h = 1, 2, \cdots, n-2, \qquad n \geqq 2.$$

Similarly

$$e_n = d_{n-2} + d_{n-4} + \cdots + d_k = T(P^{n-2}), \; k = [1 - (-1)^n]/2, \; n \geqq 2.$$

Hence by (8)

$$\begin{aligned} T(P^n) - T(P^{n-2}) &= T'_n - fT'_{n-2} + (1-f)\, TT'_{n-3} \\ &+ (1-f)\sum_h [T(P^h) - T(P^{h-2})]\, T'_{n-2-h}, \\ &\qquad (h = 2, 3, \cdots, n-2) \qquad (n \geqq 2). \end{aligned}$$

Since $T(P^2) = T'_2 - f + 1$, $T(P^3) = T'_3 + 2(1-f)T$, we have $T(P^n) = T(P^{n-2}) + K(P)$, where $K(P)$ is a polynomial in f and T. Hence $T(P^n)$ is a polynomial in f and T for every positive integer n.

4. *Conditions on* $T(N)$ *for* N *odd.* By (5), $g - T = 2S$ and $3T - g = 2R$. Hence

THEOREM 4. *If* N *is odd,* $T(N) \equiv g(N)$ (*mod* 2) *and* $g(N)/3 \leqq T(N) \leqq g(N)$.

Liouville stated this inequality for N odd and prime to 3. He instanced $N = 1$ as attaining the upper limit and $N = 5$ the lower. We prove next that if $N > 1$ the upper limit is lowered and that if N is not certain powers of certain primes the lower limit is raised.

THEOREM 5. *If* N *is odd* $T(N) \leqq g(N) - 2[E(N) + k(N)]$, *where* $E(N)$ *is the greatest integer in* $(2N/3)^{1/2}$ *and* $k(N) = 0$ *or* 1 *according as* $E(N)$ *is even or odd.*

This follows from $T = g - 2S$ and Lemmas 1 and 2.

LEMMA 1. *If* N *is odd* $S(N) \geqq 2h$, *where* h *is the number of odd solutions of* $W^2 < 2N/3$.

If W and N are odd and $W^2 < 2N/3$, then $4N - 6W^2 > 0$, $4N - 6W^2 \not\equiv 0 \pmod 4$, $4N - 6W^2 \not\equiv 10 \pmod{16}$. Hence * there are integers X, Y, Z (at least two of which are different from 0) such that $4N - 6W^2 = X^2 + 2Y^2 + 3Z^2$ and $4N = X^2 + 2Y^2 + 3Z^2 + 6W^2$. Hence X, Y, Z, W and $-X, Y, -Z, W$ are different. Now X, Y, Z, W determines $q = \sigma + \xi i + \eta H + \zeta L$ having norm N by (3), thus $\sigma = (X-Z)/2$, $\xi = (Y-W)/2$, $\eta = Z$, $\zeta = W$; and $-X, Y, -Z, W$ determines $q_1 = -\sigma + \xi i - \eta H + \zeta L$. Hence $q_1 \equiv q \pmod 2$, q and q_1 each contribute 1 to S. Finally $S \geqq 2h$.

LEMMA 2. *If* N *is odd and* h *is the number of odd solutions of*

* Dickson, *Bulletin of the American Mathematical Society*, Jan., 1927.

5

$W^2 < 2N/3$ then $2h = E(N) + k$, where $k = 0$ or 1 according as $E(N)$ is even or odd.

Let h' be the number of all solutions different from 0 of $W^2 < 2N/3$. Then $2h = h' + k$, where k is 0 or 1 according as h' is even or odd. Also $h'^2 < 2N/3 < (h'+1)^2$. Hence $h' = E(N)$.

THEOREM 6. *If $N = \prod_1^n M_h$, where n is a positive integer and every two of the M_h are relatively prime, then $g(N)/3 \leqq T(N) \leqq g(N)/2 + \frac{1}{2}\prod_1^n [g(M_h) - 2t(M_h)] < g(N) - 2t(N)$, where $t(N) = E(N) + k(N)$.*

The first two inequalities follow from theorems 1, 4 and 5. The last inequality is proved by induction.

THEOREM 7. *$T(N) > g(N)/3$ if N is odd and different from P^n, where n is odd and P is a prime of the form $6m + 5$ or $n \equiv 2 \pmod 3$ and P is a prime of the form $6m + 1$; but not conversely.*

If N is a power of 3, theorem 2 gives the result. If N is a power of a prime not excluded in the theorem, $g(N) \not\equiv 0 \pmod 3$. Then $T > g/3$ by theorem 4 since T is an integer. If N is not a power of a prime, write $N = N_1 N_2$ where $N_1 > 1$, $N_2 > 1$, and N_1 is relatively prime to N_2. The result follows by theorem 1 and the inequality $-g_h/3 \leqq 2T_h - g_h \leqq g_h$ for $h = 1, 2$ which is equivalent to that in theorem 4. The converse is not true since $T(11) = 8$, $g(11) = 12$.

Theorems 6 and 7 suggest that asymptotically there may be some simple function of $g(N)$ which gives the value of $T(N)$. Since $2T(7) = 8 = g(7)$, $2T(N) = g(N)$ if N is divisible by 7. On the other hand, for every $A \geqq 0$ there are infinitely many integers N such that $|\,2T(N) - g(N)\,| > A$. For if P is a prime of the form $4m + 1$, $g(P) = 4m + 2$ and $2T(P) \not\equiv g(P)$ since $T \equiv g \pmod 2$. Let n be any positive integer such that $2^n > A$, and $P_1, \cdots, P_n$ any n distinct primes of the form $4m + 1$. Then $|\,2T(P_h) - g(P_h)\,| > 1$ for $h = 1, \cdots, n$, and by theorem 1

$$|\,2T(N) - g(N)\,| \geqq 2^n > A \text{ if } N = \prod_1^n P_h.$$

For positive odd integers $N \leqq 75$, for $N = 3^4$, and for $N = 5^3$, the solutions of $N = x^2 + 2y^2 + 3z^2 + 6w^2$ have been found. The values of $T(N)$ thus obtained check the preceding theorems. The upper and lower limits of theorems 4, 5 and 6 coincide if $N = 5$; the upper limit is attained for $N = 1, 3, 7, 11, 17$, the lower for $N = 23$.

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Normal Ternary Continued Fraction Expansions for Cubic Irrationalities.

By P. H. Daus.

I. Introduction.

1. *Ternary Continued Fraction.* In a previous article * the expression *Normal Ternary Continued Fraction* was defined and expansions for the cube roots of integers discussed. It is the purpose of this article to discuss expansions for cubic irrationalities defined by the equation $x^3+px^2+qx+r=0$.
It will be convenient to collect here some of the definitions and results of the paper referred to above that will be needed later.

Definition. *If u_1, v_1, w_1 be any three numbers, we define a ternary continued fraction expansion for them by the equations*

$$u_{n+1}=v_n-p_nu_n;\quad v_{n+1}=w_n-q_nu_n;\quad w_{n+1}=u_n. \tag{1}$$

These equations may be written by combining them,

$$\begin{aligned} u_n &= \phantom{u_{n+1}+v_{n+1}+{}} w_{n+1}, \\ v_n &= u_{n+1} \phantom{{}+v_{n+1}} + p_n\, w_{n+1}, \\ w_n &= \phantom{u_{n+1}+{}} v_{n+1} + q_n\, w_{n+1}. \end{aligned} \tag{2}$$

The set of numbers (u_n, v_n, w_n) is called the nth complete quotient set, and (p_n, q_n) is called the nth partial quotient set. The numbers (A_n, B_n, C_n), defined by the recursion formulas (3) below form the nth convergent set to the ternary continued fraction

$$(v_1/u_1,\ w_1/u_1)=(p_1,\ q_1;\ p_2,\ q_2;\ \cdots;\ p_n,\ q_n;\ \cdots).$$

$$\begin{aligned} A_n &= q_nA_{n-1}+p_nA_{n-2}+A_{n-3}, \\ B_n &= q_nB_{n-1}+p_nB_{n-2}+B_{n-3}, \\ C_n &= q_nC_{n-1}+p_nC_{n-2}+C_{n-3}, \end{aligned} \tag{3}$$

with the initial conditions

$$\begin{vmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} .$$

2. *Relations between Complete Quotient Sets.*

* *American Journal of Mathematics,* Vol. 44 (1922), pp. 279-296. In equations (A) and (3) of this paper, p and q should be interchanged.

THEOREM I. *If u_1, v_1, w_1, any three numbers, be expanded into a ternary continued fraction, then*

$$
(4)\qquad
\begin{aligned}
u_1 &= A_{n-2}u_{n+1} + A_{n-1}v_{n+1} + A_n w_{n+1},\\
v_1 &= B_{n-2}u_{n+1} + B_{n-1}v_{n+1} + B_n w_{n+1},\\
w_1 &= C_{n-2}u_{n+1} + C_{n-1}v_{n+1} + C_n w_{n+1},
\end{aligned}
$$

and

$$
(5)\qquad
\begin{aligned}
u_{n+1} &= c_{n-2}w_1 + b_{n-2}v_1 + a_{n-2}u_1,\\
v_{n+1} &= c_{n-1}w_1 + b_{n-1}v_1 + a_{n-1}u_1,\\
w_{n+1} &= c_n\ w_1 + b_n\ v_1 + a_n\ u_1,
\end{aligned}
$$

where $a_n, b_{n-1}, \cdots$ are the cofactors of the corresponding elements of the determinant of (4).

3. *Regular Expansions.* The defining equations (1) are independent of the choice of p_n, q_n. In order to extend the continued fraction algorithm it is desirable to have the absolute values of the A's, B's and C's each form an increasing sequence of integers. Such an expansion will be called *regular,* and we will confine our attention to such expansions.

If we call $v_n/u_n = \sigma_{1,n}$ and $w_n/u_n = \sigma_{2,n}$, then the defining equations (1) may be written

$$
(6)\qquad \sigma_{1,n} = p_n + 1/\sigma_{2,n+1}\,; \quad \sigma_{2,n} = q_n + \sigma_{1,n+1}/\sigma_{2,n+1}.
$$

Jacobi * chose p_n and q_n as the greatest integers in $\sigma_{1,n}$ and $\sigma_{2,n}$, which led him into difficulties. We notice that the equation defining $\sigma_{1,n}$ is the defining equation for an ordinary continued fraction, but that for $\sigma_{2,n}$ is not. Because of this, it was found desirable to remove this limitation on q_n, and select q_n as indicated below, the purpose in view being to make the expansion regular. If $\sigma_{1,n}$ is positive, we select p_n equal to the greatest integer in $\sigma_{1,n}$. (If $\sigma_{1,n}$ is an integer p_n may be equal to or one less than $\sigma_{1,n}$.) If $\sigma_{1,n}$ is negative, then $|\,p_n\,|$ is selected equal to or one more than the greatest integer in $|\,\sigma_{1,n}\,|$. The essential point of this selection is to make $|\,u_{n+1}\,| < |\,u_n\,|$. In either case q_n is selected arbitrarily, usually approximately equal to $\sigma_{2,n}$, subject to the limitation that the expansion be regular. We give as an illustration two examples, showing the reduction and rebuilding of (A_n, B_n, C_n).

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* Jacobi, C. G. J. *Gesammalte Werke*, Vol. VI, pp. 385-426.

Example 1.

u	v	w	$p, q;$	A	B	C
				1	0	0
				0	1	0
				0	0	1
17	44	13	2, 0;	1	2	0
10	13	17	1, 1;	1	3	1
3	7	10	2, 1;	3	7	2
1	7	3	6, 1;	10	27	8
1	2	1	2, 1;	17	44	13
0	0	1				

It will be noted that q_3 and q_4 are less than the greatest integer in the corresponding σ's. The selection in this particular example was guided by the next theorem.

Example 2.

u	v	w	$p, q;$	A	B	C
				1	0	0
				0	1	0
				0	0	1
9	—3	—53	0,—6;	1	0	—6
—3	1	9	0,—3;	—3	1	—18
1	0	—3	0,—3;	9	—3	—53
0	0	1				

This example illustrates a case when the σ's are negative.

4. *Skew-palindromic Expansions.* An important case of a regular expansion (illustrated by the examples above) is one which may be considered as an extension of an ordinary continued fraction in which the partial quotients are palindromic, that is, read the same backward as forward. In our case, however, we define a *skew-palindromic* expansion as one in which the q's are palindromic and the p's omitting the first, are also palindromic.

THEOREM II. *If u_1, v_1, w_1, three integral rational numbers, relatively prime, be expanded into a ternary continued fraction, then the necessary and sufficient condition that the sequence $u_1, u_2, \cdots, u_n$, be the sequence A_n, $A_{n-1}, \cdots, A_1$, $(A_k = u_{n-k+1})$, where (A_n, B_n, C_n) is the last convergent set, is that if we omit the first partial quotient set, the balance of the expansion be skew-palindromic.*

For such an expansion, it is evident from the manner in which the p's are selected that $|u\text{'s}|$ form a decreasing sequence and consequently the $|A\text{'s}|$ an increasing sequence of integers. From the properties of convergents it will later be shown that for the type of expansion we are to discuss, that $|B\text{'s}|$ and $|C\text{'s}|$ also form an increasing sequence of integers. In the case the p's and q's are always positive, this is evident from the form of the recursion formulas (3).

II. NORMAL TERNARY CUBIC FORMS.

5. *Normal Cubic.* The discussion of ternary continued fractions is closely allied to the problem of determining the units in a given domain, defined by a root θ, of the irreducible equation

$$x^3 + px^2 + qx + r = 0. \tag{7}$$

That is, we seek integers x, y, z such that the norm of $(x + \theta y + \theta^2 z)$ will be unity. If we indicate the three roots of (7) by $\theta, \theta', \theta''$, then

$$N(x + \theta y + \theta^2 z) = (x + \theta y + \theta^2 z)(x + \theta' y + \theta'^2 z)(x + \theta'' y + \theta''^2 z). \tag{8}$$

This expression, when multiplied out and simplified by means of the well-known relations between the roots, will be called the *normal ternary cubic* and denoted by $N(x, y, z)$. It becomes

$$\begin{aligned} N(x, y, z) = x^3 - px^2y + qxy^2 - ry^3 + (p^2 - 2q)x^2z + (q^2 - 2pr)xz^2 \\ + pry^2z - qryz^2 + r^2z^3 + (3r - pq)xyz. \end{aligned} \tag{9}$$

In the special case of the reduced cubic $p = 0$ and

$$N(x, y, z) = x^3 + qxy^2 - ry^3 - 2qx^2z + q^2xz^2 - qryz^2 + r^2z^3 + 3rxyz. \tag{9$'$}$$

In the further special case $p = 0$, $r = -D$, so that $x^3 = D$, we have the form of the so-called, misnamed, Pellian cubic,

$$N(x, y, z) = x^3 + Dy^3 + D^2z^3 - 3Dxyz. \tag{9$''$}$$

For convenience in computation, it is highly desirable to have equation (9) in determinant form. To that end, we consider the automorphs of $N(x, y, z)$.*

Let x, y, z be a solution of $N(x, y, z) = 1$, and let $(x_1 + \theta y_1 + \theta^2 z_1)$ and $(x_2 + \theta y_2 + \theta^2 z_2)$ be two numbers, such that

$$x_2 + \theta y_2 + \theta^2 z_2 = (x + \theta y + \theta^2 z)(x_1 + \theta y_1 + \theta^2 z_1). \tag{10}$$

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* See H. W. Tanner, "Notes on a Ternary Cubic," *Proceedings of the London Mathematical Society*, Vol. 10 (1895), p. 187.

Multiplying out and equating coefficients of like irrationalities and reducing by means of (7), we have

$$(11)\qquad \begin{aligned} x_2 &= xx_1 - rzy_1 + (-ry + rpz)z_1, \\ y_2 &= yx_1 + (x - qz)y_1 + (-rz - qy + pqz)z_1, \\ z_2 &= zx_1 + (y - pz)y_1 + (x - qz - py + p^2z)z_1. \end{aligned}$$

Now equation (10) is equivalent to

$$(10')\qquad N(x_2, y_2, z_2) = N(x, y, z)N(x_1, y_1, z_1).$$

Interpreting (11) as a transformation on the form $N(x, y, z)$, we have

$$N(x,y,z) \begin{bmatrix} x & -rz & -r(y-pz) \\ y & x-qz & -rz-q(y-pz) \\ z & y-pz & x-qz-p(y-pz) \end{bmatrix} N(x,y,z),$$

where x, y, z is a solution of $N(x, y, z) = 1$. Applying this transformation to the right hand member of $N(x, y, z) = 1$ itself, we see

$$(12)\qquad N(x,y,z) = \begin{vmatrix} x & -rz & -r(y-pz) \\ y & x-qz & -rz-q(y-pz) \\ z & y-pz & x-qz-p(y-pz) \end{vmatrix}.$$

The corresponding special cases become

$$(12')\qquad N(x,y,z) = \begin{vmatrix} x & -rz & -ry \\ y & x-qz & -rz-qy \\ z & y & x-qz \end{vmatrix},$$

and

$$(12'')\qquad N(x,y,z) = \begin{vmatrix} x & Dz & Dy \\ y & x & Dz \\ z & y & x \end{vmatrix}.$$

6. *Normal Cubic Surface.* Let us now consider some elementary geometrical properties of the surface $N(x, y, z) = m$.* Let us denote the expression $x + \theta y + \theta^2 z$ by U and the corresponding expressions in θ' and θ'' by U' and U''. Our surface is then

$$(13)\qquad U\,U'\,U'' = m.$$

It is apparent that the surface has three binodes in the plane at infinity in the direction given by the intersection of the asymptotic planes $U = 0$, $U' = 0$, $U'' = 0$, taken in pairs. There are two cases to consider. First, when equation (7) has but one real root and, secondly, when it has three real roots.

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* See H. W. Tanner, *loc. cit.* For further details see W. E. H. Berwick, *Proceedings of the London Mathematical Society*, (2) Vol. 12 (1913), p. 393-429.

In the first case, only one of the asymptotic planes is real, say $U=0$. The real line of intersection of the imaginary planes becomes an asymptotic line. If we cut the surface, $N(x,y,z)=m$, by the plane $U=k$, we get a section given by $U'U''=m/k$. We readily recognize that the section is an ellipse which decreases continuously in size as the cutting plane moves further away from the origin, gradually closing in on the line $U'=0$, $U''=0$. The surface has a spur in the direction of this line, which grows larger as we progress from infinity towards the origin, and finally spreads out over the plane $U=0$, the entire surface lying on one side of this plane. The surface may be visualized by a deformation of the canonical form $(x^2+y^2)z=m$, where $z=0$ plays the rôle of $U=0$, and the z-axis that of the asymptotic line.

In the second case, when all the roots are real, the three asymptotic planes are real and divide space into eight regions. The surface consists of four sheets, no two of which lie in adjacent regions. Every section by a plane through an intersection corresponds to the canonical form $xy^2=c$, while the section parallel to one of the planes is a hyperbola. This surface can be visualized by a deformation of the canonical form $xyz=m$.

7. *Direction Cosines of Asymptotic Line.* The direction cosines of the line of intersection of the planes $U'=0$, $U''=0$ are proportional to the determinants formed from the matrix

$$\begin{Vmatrix} 1 & \theta' & \theta'^2 \\ 1 & \theta'' & \theta''^2 \end{Vmatrix} .$$

This gives, where λ, μ, ν are the direction cosines,

$$\lambda:\mu:\nu=\theta'\theta''^2-\theta''\theta'^2 : -(\theta''^2-\theta'^2) : \theta''-\theta', \tag{14}$$
$$\lambda:\mu:\nu=N(\theta)/\theta : p+\theta : 1=q+p\theta+\theta^2 : p+\theta : 1.$$

If we call the intersection of the planes an asymptotic line, we see that its direction cosines are given by (14) and the direction cosines $\lambda':\mu':\nu'$ and $\lambda'':\mu'':\nu''$ of the other two asymptotic lines are given by replacing θ in (14) by θ' and θ'', respectively.

It is apparent that in the first case we should expect only one expansion and in the second case three expansions. The results of this last paragraph suggest that we select as u_1, v_1, w_1, the numbers $1, p+\theta, q+p\theta+\theta^2$. It is readily shown that under the conditions we have set for the selection of p_n and q_n, the partial quotients after p_1, q_1, p_2 are exactly the same as if we started with $1, \theta, \theta^2$. This can be seen by comparing the following expansions. If the expansion for $1, \theta, \theta^2$ is

(15) $$(p_1, q_1;\ p_2, q_2;\ p_3, q_3;\ \cdots),$$

then the expansion for $1,\ p+\theta,\ q+p\theta+\theta^2$ is

(15′) $$(p_1+p, q+pp_1+q_1;\ p+p_2, q_2;\ p_3, q_3;\ \cdots),$$

as may be readily verified by writing out the complete quotient sets. In fact, we might have started with $1,\ l+\theta,\ m+n\theta+\theta^2$, ($l, m, n$, integers) and obtained the same expansion except for p_1, q_1 and p_2 The selection of $1,\ p+\theta,\ q+p\theta+\theta^2$ has the advantage that the convergents (A_n, B_n, C_n) provide us directly with the solution of $N(x, y, z)=1$.

III. Periodic Expansions.

8. *Periodic Expansions.* After these preliminary considerations, we come to the following theorem, which is fundamental in the consideration of periodic expansions.

Theorem III. *If in a ternary continued fraction, the expansion becomes periodic after a finite number of terms, then $\sigma_{1,1}$ and $\sigma_{2,1}$ are roots of cubic equations with rational coefficients.**

As explained in the last section, we will confine our attention to the case where $\sigma_{1,1}=p+\theta$; $\sigma_{2,1}=q+p\theta+\theta^2$, θ being a real root of the irreducible cubic

(7) $$x^3+px^2+qx+r=0.$$

It will be convenient to use as well as the complete quotient set (u_n, v_n, w_n), composed of linear functions of θ and θ^2, the rationalized complete quotient set $(\bar{u}_n, \bar{v}_n, \bar{w}_n)$, defined as follows.

$\bar{u}_n=\alpha_n=$ Norm of $u_n=u_nu_n'u_n''$, u_n' and u_n'' being the conjugates of u_n.

(16) $$\begin{aligned}\bar{v}_n&=\alpha_n'+\beta_n'\theta+\gamma_n'\theta^2=v_nu_n'u_n''.\\ \bar{w}_n&=\alpha_n''+\beta_n''\theta+\gamma_n''\theta^2=w_nu_n'u_n''.\end{aligned}$$

$\bar{u}, \bar{v}, \bar{w}$ may have a common factor, which is not to be removed as in Jacobi's work. In this connection compare the illustrative example below.

We compute the rationalized quotient sets in order as follows. Suppose $\bar{u}_n, \bar{v}_n, \bar{w}_n$ equal to $u_nu_n'u_n''$, $v_nu_n'u_n''$, $w_nu_n'u_n''$ respectively have been found. Then before rationalization

$$\begin{aligned}u_{n+1}:u_{n+1}:w_{n+1}&=(v_n-p_nu_n)u_n'u_n'':(w_n-q_nu_n)u_n'u_n'':u_nu_n'u_n''\\ &=u_{n+1}u_n'u_n'':v_{n+1}u_n'u_n'':w_{n+1}u_n'u_n''.\end{aligned}$$

* For proof, see D. N. Lehmer, *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 360-364, or my previous article.

Now to rationalize $u_{n+1}u_n'u_n''$, we multiply by the conjugates of each factor, namely the product $u'_{n+1}u''_{n+1}u_nu_n''u_nu_n'$. This gives, since $\bar{u}_n = \alpha_n$,

$$u_{n+1} : v_{n+1} : w_{n+1} = u_{n+1}u'_{n+1}u''_{n+1}\alpha^2 : v_{n+1}u'_{n+1}u''_{n+1}\alpha^2 : w_{n+1}u'_{n+1}u''_{n+1}\alpha^2.$$

We notice that we obtain $\bar{u}_{n+1}$, $\bar{v}_{n+1}$, $\bar{w}_{n+1}$, by dividing out α^2. A numerical example will perhaps make this clear.

9. *Numerical Example.* In the expansion of the real root of $x^3 + x - 5 = 0$, we find that $\bar{u}_2 = 3$, $\bar{v}_2 = 1 + 2\theta - \theta^2$, $\bar{w}_2 = 2 + \theta + \theta^2$, $p_2 = 0$ and select $q_2 = 1$. Consequently $u_3 : v_3 : w_3 = 1 + 2\theta - \theta^2 : -1 + \theta + \theta^2 : 3$. The product of the conjugates of $1 + 2\theta - \theta^2$, viz. $(1 + 2\theta' - \theta'^2)(1 + 2\theta'' - \theta''^2)$, when reduced by means of the elementary symmetric functions of the roots is $3(6 + \theta + 2\theta^2)$, so that

$$\begin{aligned} u_3 : v_3 : w_3 &= 3\,(6 + \theta + \theta^2)\,(1 + 2\theta - \theta^2) : 3(6 + \theta + 2\theta^2)\,(-1 + \theta + \theta^2) \\ &\qquad : 9(6 + \theta + 2\theta^2) \\ &= 9 \cdot 7 : 9(3 + 4\theta + \theta^2) : 9(6 + \theta + 2\theta^2). \end{aligned}$$

So that

$$\bar{u}_3 = 7;\ \bar{v}_3 = 3 + 4\theta + \theta^2;\ \bar{w}_3 = 6 + \theta + 2\theta^2.$$

Below appears a convenient arrangement of the expansion, showing the convergents and the value of $N(x, y, z) = m$, using equation (12′) with $x = C_n$, $y = B_n$, $z = A_n$.

Expansion for $x^3 + x - 5 = 0$

$\bar{u} = \alpha$	$\bar{v}$	$\bar{w}$	$p, q;$	A	B	C	m
				1	0	0	
				0	1	0	
				0	0	1	
1	θ	$1 + \theta^2$	1, 3;	1	1	3	5
3	$1 + 2\theta - \theta^2$	$2 + \theta + \theta^2$	0, 1;	1	2	3	9
7	$3 + 4\theta + \theta^2$	$6 + \theta + 2\theta^2$	1, 1;	2	3	7	3
9	$6 + 3\theta$	$3 + 3\theta + 3\theta^2$	1, 1;	4	6	13	1
1	$2 + \theta$	$2 + 2\theta + 2\theta^2$	3, 4;	23	35	76	9
3	$6 + 3\theta$	$2 + 2\theta + 2\theta^2$	3, 1;	37	56	122	7
9	$8 + 4\theta + \theta^2$	$6 + 3\theta + 3\theta^2$	1, 1;	64	97	211	3
5	$5 + 2\theta - \theta^2$	$\theta + 2\theta^2$	1, 1;	124	188	409	1
1	θ	$3 + \theta + \theta^2$	1, 6;		. . .		
3	$4 + 2\theta - \theta^2$	$2 + \theta + \theta^2$	1, 1;				
7	$3 + 4\theta + \theta^2$	$6 + \theta + 2\theta^2$					
	. . .						

It may be well to point out several characteristics of this expansion, which will be the subject of discussion in the following pages. The expansion is periodic, and has one non-recurring q and two non-recurring p's. If we omit the first partial quotient set and the last two sets of the period, the balance is skew-palindromic. The values of m form the same sequence as the values of α, in the reverse order. The convergent set, two before the close of the period gives us a unit, but in this case it is not a fundamental unit. Indeed it is easily verified that $(409 + 188\theta + 124\theta^2) = (13 + 6\theta + 4\theta^2)^2$. There are certain other numerical relations which involve the last two partial quotient sets and the non-recurring elements, but these will be pointed out later.

10. *Periodicity of m_k.* We will now establish a theorem which we need in the consideration of the values of the normal cubic obtained from the convergent sets.

THEOREM IV. *If 1, $p + \theta$, $q + p\theta + \theta^2$, be expanded into a ternary continued fraction, which ultimately becomes periodic, and if m_k is the value of $N(C_k, B_k, A_k)$, then the sequence m_k also becomes periodic.*

To prove this we will establish certain lemmas, which we also need in other connections.

LEMMA I. *If 1, $p + \theta$, $q + p\theta + \theta^2$, be expanded into a ternary continued fraction, then the value of m_k is*

$$(17)\qquad m_k = \frac{(\beta' - p\gamma')^3 + p(\beta' - p\gamma')^2\gamma' + q(\beta' - p\gamma')\gamma'^2 + r\gamma'^3}{(\beta'\gamma'' - \beta''\gamma')^2}\,\alpha_{k+1}.$$

(The subscript for β', γ', β'', γ'' is $k + 1$, but has been omitted for convenience).

From the lemma it will follow, that if the expansion ultimately becomes periodic, then the values of m_k also become periodic.

We will write instead of $A_k, A_{k-1}, \cdots, B_k, \cdots, A_0, A_1, \cdots, B_0, \cdots$ and omit the subscript $k + 1$ on $\alpha, \beta, \cdots$. Using this notation, we may obtain by using Theorem I, equations (4), the following six equations*

$$(18)\qquad \begin{aligned} (C_0 - qA_0)\alpha'' + (C_1 - qA_1)\alpha' + (C_2 - qA_2)\alpha &= -rB_0\gamma'' - rB_1\gamma', \\ (B_0 - pA_0)\alpha'' + (B_1 - pA_1)\alpha + (B_2 - pA_2)\alpha &= -rA_0\gamma'' - rA_1\gamma', \\ A_0\,\alpha'' + A_1\,\alpha' + A_2\,\alpha &= C_0\gamma'' + C_1\gamma'. \end{aligned}$$

* For the method see the original paper, *loc. cit.*, pp. 286-289.

$$
\begin{aligned}
&A_0\beta'' + A_1\beta' = && B_0\gamma'' + && B_1\gamma', \\
(19)\quad &B_0\beta'' + B_1\beta' = && (C_0 - qA_0 + pB_0)\gamma'' + && (C_1 - qA_1 + pB_1)\gamma', \\
&C_0\beta'' + C_1\beta' = && (-rA_0 + pC_0)\gamma'' + && (-rA_1 + pC_1)\gamma'.
\end{aligned}
$$

By eliminating α', α'' and by means of certain algebraic manipulations, we obtain the desired equation (17).

11. *Reciprocal Units.* Equation (17) can be put into a simpler form due to Lemma II, which also gives us information concerning the reciprocal of a given unit.

LEMMA II. *If* $1, p+\theta, q+p\theta+\theta^2$, *be expanded into a ternary continued fraction, then* $\alpha = \beta'\gamma'' - \gamma'\beta''$ *for all values of the subscript.*

Equation (17) takes the simpler form

$$(17')\qquad \alpha_{k+1}m_k = \delta^3{}_{k+1} + p\delta^2{}_{k+1}\gamma'{}_{k+1} + q\delta_{k+1}\gamma'^2{}_{k+1} + r\gamma'^3{}_{k+1}, \quad (\delta = \beta' - p\gamma').$$

The proof of the lemma follows. From equations (5), adopting the abbreviated notation,

$$
(20)\qquad
\begin{aligned}
u_{k+1} &= a_2 + b_2(p+\theta) + c_2(q+p\theta+\theta^2), \\
v_{k+1} &= a_1 + b_1(p+\theta) + c_1(q+p\theta+\theta^2), \\
w_{k+1} &= a_0 + b_0(p+\theta) + c_0(q+p\theta+\theta^2).
\end{aligned}
$$

Let $u'_{k+1}u''_{k+1} = L + M\theta + N\theta^2$, so that

(21)

$$
\begin{aligned}
\alpha &= [(a_2 + pb_2 + qc_2) + (b_2 + pc_2)\theta + c_2\theta^2]\,(L + M\theta + N\theta^2), \\
\alpha' + \beta'\theta + \gamma'\theta^2 &= [(a_1 + pb_1 + qc_1) + (b_1 + pc_1)\theta + c_1\theta^2]\,(L + M\theta + N\theta^2), \\
\alpha'' + \beta''\theta + \gamma''\theta^2 &= [(a_0 + pb_0 + qc_0) + (b_0 + pc_0)\theta + c_0\theta^2]\,(L + M\theta + N\theta^2).
\end{aligned}
$$

Multiplying the last two of these equations out and comparing coefficients, we have

$$
(22)\qquad
\begin{aligned}
\beta' &= L(b_1 + pc_1) + M(a_1 + pb_1) - N(b_1q + c_1r), \\
\beta'' &= L(b_0 + pc_0) + M(a_0 + pb_0) - N(b_0q + c_0r), \\
\gamma' &= Lc_1 + Mb_1 + Na_1, \\
\gamma'' &= Lc_0 + Mb_0 + Na_0.
\end{aligned}
$$

Now form the product $\beta'\gamma'' - \gamma'\beta''$. It can be written in the form

$$
\begin{aligned}
\beta'\gamma'' - \gamma'\beta'' &= (b_1c_0 - b_0c_1)(L^2 - qLN + rMN) \\
&\quad + (a_0c_1 - a_1c_0)(-rN^2 - LM + pLN) \\
&\quad + (a_1b_0 - a_0b_1)(M^2 - pMN - LN + qN^2) \\
&\equiv R(b_1c_0 - b_0c_1) + S(a_0c_1 - a_1c_0) + T(a_1b_0 - a_0b_1).
\end{aligned}
$$

In determinant form

$$(23)\qquad \beta'\gamma'' - \gamma'\beta'' = \begin{vmatrix} R & S & T \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{vmatrix}.$$

The values of L, M, N can be found from the equation which defines them, and the elementary symmertic functions. They are

$$(24)\qquad \begin{aligned} L &= a_2^2 + pa_2b_2 + rb_2c_2 + qb_2^2, \\ M &= -(a_2b_2 + pa_2c_2 + rc_2^2 + qb_2c_2), \\ N &= b_2^2 - a_2c_2. \end{aligned}$$

Now α may be expressed in determinant form by putting

$$x = a_2 + pb_2 + qc_2, \quad y = b_2 + pc_2, \quad z = c_2,$$

in the general form of $N(x + \theta y + \theta^2 z)$, equation (12).
This gives

$$(25)\qquad \alpha = \begin{vmatrix} a_2 + pb_2 + qc_2 & -rc_2 & -rb_2 \\ b_2 + pc_2 & a_2 + pb_2 & -rc_2 - qb_2 \\ c_2 & b_2 & a_2 \end{vmatrix}.$$

In this form, we observe that L, M, N are the cofactors of the elements of the first row of (25). If now we replace each element of (25) by its cofactor, expressed in terms of L, M, N, we get, using a well-known theorem on determinants,

$$(26)\qquad \alpha^2 = \begin{vmatrix} L & M & N \\ -rN & L - qN & M - pN \\ -r(M - pN) & (pq - r)N - qM & L + (p^2 - q)N - pN \end{vmatrix}.$$

Also, the cofactor of any element of (26) is α times the corresponding element of (25). Consequently, forming the cofactors of the elements of the third row of (26), we obtain

$$(27)\qquad \begin{aligned} M^2 - pMN - LN + qN^2 &\equiv T = c_2\alpha, \\ -rN^2 - LM + pLN &\equiv S = b_2\alpha, \\ L^2 - qNL + rMN &\equiv R = a_2\alpha. \end{aligned}$$

It follows by substituting these values of R, S, T in (23), that

$$\beta'\gamma'' - \beta''\gamma' = \alpha \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{vmatrix} = \alpha \cdot 1 = \alpha.$$

This completes the proof of the lemma, establishes equation (17′), and incidentally gives the following method for determining the reciprocal unit.

Suppose now that $$N(x, y, z) = 1, \quad \text{and}$$
$$(x + \theta y + \theta^2 z)(L + M\theta + N\theta^2) = 1, \quad \text{so that}$$
$N(L, M, N)$ is also 1, and L, M, N is the reciprocal solution corresponding to x, y, z. Write $N(x, y, z)$ in determinant form.

$$\begin{vmatrix} x & -rz & -r(y-pz) \\ y & x-qz & -rz-q(y-pz) \\ z & y-pz & (x-qz)-p(y-pz) \end{vmatrix} = 1.$$

Then L, M, N are the cofactors of the elements of the first row. This is a convenient device often used in numerical calculations.

12. *Convergence.* The periodicity of the sequence m_k, and the geometry of the surface $N(x, y, z) = m$ afford a proof that B_k/A_k and C_k/A_k actually converge towards $p + \theta$ and $q + p\theta + \theta^2$ respectively, if the expansion is regular.

Consider the line which is the intersection of two of the asymptotic planes of the surface $N(x, y, z) = m$.

$$x/(q + p\theta + \theta^2) = y/(p + \theta) = z/1.$$

Let (C_k, B_k, A_k) be any point on the surface $N(x, y, z) = m$, where m is one of the values assumed by m_k. The distance d_k, from this point to the line is given by

$$(28) \quad d_k^2 = (C_k^2 + B_k^2 + A_k^2)\left[1 - \frac{(C_k w_1 + B_k v_1 + A_k)^2}{(C_k^2 + B_k^2 + A_k^2)(w_1^2 + v_1^2 + 1)}\right],$$

where $w_1 = q + p\theta + \theta^2$ and $v_1 = p + \theta$. This formula can be written in the form

$$(29) \quad \begin{aligned} &(w_1^2 + v_1^2 + 1)d_k^2 \\ &= (B_k - A_k v_1)^2 + (C_k - A_k w_1)^2 + (C_k v_1 - B_k w_1)^2. \end{aligned}$$

Now as (C_k, B_k, A_k) take on successive values which are one period apart and all solutions of $N(x, y, z) = m$, the value of $|z| = |A_k|$ continually increases by the assumption of a regular expansion. The points on the surface approach continually nearer to the asymptotic line, so that $d_k \to 0$. Since v_1 and w_1 are finite, it follows that the $|B\text{'s}|$ and $|C\text{'s}|$ each form an increasing sequence if the $|A\text{'s}|$ do, such that $B_k - A_k v_1$, $C_k - A_k w_1$, $Cv_1 - B_k w_1$, all approach zero as a limit. This result we state as

THEOREM V. *If* $1, p+\theta, q+p\theta+\theta^2$, *be expanded into a regular continued fraction, which ultimately becomes periodic, then*

$$\lim_{k\to\infty} B_k/A_k = p+\theta, \quad \textit{and} \quad \lim_{k\to\infty} C_k/A_k = q+p\theta+\theta^2.$$

IV. NORMAL TERNARY CONTINUED FRACTIONS.

13. *Determination of Unit.* We will use theorem IV and the corresponding equation (17′) to prove the following theorem.

THEOREM VI. *If* $1, p+\theta, q+p\theta+\theta^2$, *be expanded into a periodic continued fraction, such that the q's after the first are periodic and the p's after the second are periodic, and if the period contain n sets of partial quotients, then* (C_n, B_n, A_n) *is a solution of* $N(x, y, z) = 1$.

Under the conditions of the hypothesis,

$$(30) \qquad u_{k+2} : v_{k+2} : w_{k+2} = u_{k+n+2} : v_{k+n+2} : w_{k+n+2}, \qquad (k = 1, 2, \cdots, \infty).$$

However, in general

$$(31) \qquad w_{k+2} = u_{k+1} \quad \text{and} \quad w_{k+n+2} = u_{k+n+1}.$$

By successive combinations of these two sets of equations we find

$$(32) \qquad \begin{aligned} u_{n+k+2} &= \lambda u_{k+2}, & (k &= 0, 1, \cdots, \infty), \\ v_{n+k+2} &= \lambda v_{k+2}, & (k &= 1, 2, \cdots, \infty), \\ w_{n+k+2} &= \lambda w_{k+2}, & (k &= 1, 2, \cdots, \infty), \end{aligned}$$

where

$$\lambda = u_{n+2}/u_2.$$

By referring to equations (4) of theorem I, and making the necessary changes in subscripts, we have

$$A_{n+2}w_{n+3} + A_{n+1}v_{n+3} + A_n u_{n+3} = 1,$$

or

$$\lambda(A_{n+2}w_3 + A_{n+1}v_3 + A_n u_3) = 1.$$

Therefore λ is a unit in the domain $K(\theta)$, that is, $N(\lambda) = 1$. Furthermore, from (32), $N(u_{k+2}) = \alpha_{k+2}$ forms a periodic sequence beginning with $k = 0$, or $\alpha_2, \alpha_3, \cdots$ is a periodic sequence of period n.

Now let us impose the condition of the hypothesis $q_{n+2} = q_2$. From the equations (2)

$$\begin{aligned} w_{n+2} &= v_{n+3} + q_{n+2}\, u_{n+2}, \\ w_2 &= v_3 \quad + q_2 \quad u_2. \end{aligned}$$

Since $w_2 = 1$, and $u_{n+1} = w_{n+2}$, and $q_2 = q_{n+2}$, we find from (32)

$$u_{n+1} = \frac{w_{n+2}}{w_2} = \frac{\lambda v_3 + \lambda u_2 q_2}{v_3 + u_2 q_2} = \lambda,$$

or

(33) $$N(u_{n+1}) = \alpha_{n+1} = 1.$$

Consequently the α's form a purely periodic sequence. Equation (17′) now becomes, for the special case under consideration

(34) $$m_n = (\beta'_{n+1} - p\gamma'_{n+1})^3 + p(\beta'_{n+1} - p\gamma'_{n+1})^2\gamma' + q(\beta'_{n+1} - p\gamma'_{n+1})\gamma'^2_{n+1} + r\gamma'^3_{n+1}.$$

By definition

$$v_{n+1} = u_{n+2} + p_{n+1}u_{n+1} = u_{n+1}(u_2 + p_{n+1}),$$

or

(35) $$v_{n+1}/u_{n+1} = p_{n+1} + p - p_1 + \theta,$$

since

$$u_{n+2} = \lambda u_2,\ u_{n+1} = \lambda, \text{ and } u_2 = p - p_1 + \theta.$$

As

$$\alpha'_{n+1} + \beta'_{n+1}\theta + \gamma'_{n+1}\theta^2 = \frac{v_{n+1}}{u_{n+1}} N(u_{n+1}),$$

it follows that

(36) $$\alpha'_{n+1} = p_{n+1} - p_1 + p;\ \beta'_{n+1} = 1;\ \gamma'_{n+1} = 0.$$

These last two values, when substituted in (34) give $m_n = 1$, or

$$N(C_n, B_n, A_n) = 1,$$

proving the theorem.

By a method similar to that used to find (36), we can find α''_{n+1}, β''_{n+1}, γ''_{n+1}, and as we will need them later, we evaluate them now.

$$w_{n+1} = v_{n+2} + q_{n+1}u_{n+1} = u_{n+3} + p_{n+2}u_{n+2} + q_{n+1}u_{n+1}.$$
$$w_{n+1}/u_{n+1} = u_3 + p_{n+2}u_2 + q_{n+1}.$$

Now

$$u_2 = -p_1 + p + \theta,$$
$$u_3 = (p_1 - p)p_2 - q_1 + q + (p - p_2)\theta + \theta^2,$$

and

$$\alpha''_{n+1} + \beta''_{n+1}\theta + \gamma''_{n+1}\theta^2 = \frac{w_{n+1}}{u_{n+1}} N(u_{n+1}).$$

Hence

(37) $$\begin{aligned} \alpha''_{n+1} &= (p_1 - p)(p_2 - p_{n+2}) + q - q_1 + q_{n+1}, \\ \beta''_{n+1} &= p - p_2 + p_{n+2}, \\ \gamma''_{n+1} &= 1. \end{aligned}$$

14. *Special Cases.* It should be pointed out that the essential points of the hypothesis of theorem VI are that the q's are periodic after the first, and the p's are periodic after the second. No limitations are imposed upon q_1, p_1 and p_2, and in special cases these might be such that the q's are purely periodic, and the p's periodic after p_1 or even purely periodic. For example, it will be shown later, or may easily be verified directly, that if θ is a root of $x^3+qx-1=0$, then the expansion for $(1, \theta, q+\theta^2)$ is $(0, q; 0, q; \cdots)$, that is, purely periodic with $m=1$. However, it may be interpreted as an expansion, which satisfies the hypothesis, so that $m_1=1$. Again, it is found that if θ is the real root of $x^3+5x-3=0$, then

$$(1, \theta, 5+\theta^2) = (0, 4; 2, 1; 2, 1; 5, 1; 2, 1; 2, 3; 2, 1; 2, 1; 5, 1; \cdots)$$

This expansion has 5 terms in the period, and may be considered as one which is periodic after the first partial quotient set. However, for the puropose of the theorem, we may consider that the q's after the first and the p's after the first and second are periodic, and hence $N(C_5, B_5, A_5)=1$.

Let us further consider the numerical example previously given, $x^3+x-5=0$. It will be noticed that the part of the expansion obtained by deleting p_1, q_1; from the beginning and p_{n+1}, q_{n+1}; p_{n+2}, q_{n+2}; from the end is skew-palindromic. Consequently (theorem II), if the unit is expanded into a finite continued fraction using the same set of partial quotients, the u's will be the same sequence as the A's in reverse order. This is the key to experimental attempts to find expansions, for having first found a unit (not necessarily the fundamental one), we seek a finite expansion corresponding to it, which, after p_1, q_1; is skew-palindromic.

It should be noted that the period of partial quotient sets for the desired periodic expansion is

$$p_3, q_3;\ p_4, q_4;\ \cdots\ p_n, q_n;\ p_{n+1}, q_{n+1};\ p_{n+2}, q_2;$$

while the $N(C_n, B_n, A_n)=1$. That is, the expansion of the unit does not depend upon $p_{n+1}, p_{n+2}, q_{n+1}$. These numbers are involved in equations (36) and (37) and just what rôle they play in the expansion will be indicated in the next definition and theorem.

15. *Normal Ternary Continued Fractions.* The experimental determination of a large number of periodic expansions led to the following definition of a *Normal expansion.*

If $1, p+\theta, q+p\theta+\theta^2$, *expands into a regular periodic ternary continued fraction with n partial quotient sets in the period, such that the q's are periodic after the first and the p's after the second, and if that part of*

the expansion indicated by $p_2, q_2; p_3, q_3; \cdots; p_n, q_n;$ *is skew-palindromic, and further if* $p_{n+1} = p_{n+2} = p_1 + p_2 - p$ and $q_{n+1} = (p_1 - p)^2 + 2q_1 - q$, *then the expansion is normal.*

This definition was suggested by the following important theorem of which the last is a special case.

THEOREM VII. *If* $1,\ p + \theta,\ q + p\theta + \theta^2$ *be expanded into a normal ternary continued fraction, then* $(C_{n-k+1}, B_{n-k+1}, A_{n-k+1})$ *is a solution of the normal cubic* $N(x, y, z) = \alpha_k$, $(k = 1, 2, \cdots, n)$.

The proof is long and we omit the details.* Under the assumptions that

$$p_{k+2} = p_{n-k+1}, \quad q_{k+1} = q_{n-k}, \qquad (k = 1, 2, \cdots, n), \tag{38}$$

$$p_{n+1} = p_{n+2} = p_1 + p_2 - p, \quad q_{n+1} = (p_1 - p)^2 + 2q_1 - q, \tag{39}$$

we first show by mathematical induction that

$$u_k = u_{n+1}\,(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2). \tag{40}$$

To establish this for the special cases of $k = 1, 2, 3$, use is made of equations (18) and (19) and the special values of $\beta'_{n+1} = 1$, $\gamma'_{n+1} = 0$, $\gamma''_{n+1} = 1$, $\alpha_{n+1} = 1$, and the values of α''_{n+1}, α'_{n+1}, β''_{n+1} given by (36) and (37). Since $N(u_{n+1}) = 1$, the theorem readily follows.

16. *Fundamental Units.* It should be pointed out that n may be interpreted as the number of terms in one period or any number of periods, for if the expansion is normal, it is normal whether we consider only one period or m periods. That means, that instead of (40) we may write

$$u_k = u_{mn+1}\,(C_{mn-k+1} + B_{mn-k+1}\,\theta + A_{mn-k+1}\,\theta^2), \tag{41}$$

or in particular

$$u_1 = 1 = u_{mn+1}\,(C_{mn} + B_{mn}\theta + A_{mn}\theta^2). \tag{42}$$

Now by successive applications of the first equation of (32) and the value of $\lambda = u_{n+1}$, it readily follows that

$$u_{mn+1} = u^m{}_{n+1}. \tag{43}$$

Hence we have the two equations

$$\begin{aligned} u_1 &= u_{n+1}\,(C_n + B_n\theta + A_n\theta^2) = 1, \\ u_1 &= u^m{}_{n+1}\,(C_{mn} + B_{mn}\theta + A_{mn}\theta^2) = 1. \end{aligned} \tag{44}$$

Consequently,

$$(C_{mn} + B_{mn}\theta + A_{mn}\theta^2) = (C_n + B_n\theta + A_n\theta^2)^n. \tag{45}$$

* For the method see the original paper, *loc. cit.*, pp. 292-294.

This proves that the unit obtained from the mth period of the expansion is the mth power of that obtained from the first period. If $(C_n + B_n\theta + A_n\theta^2)$ is a fundamental unit, say ϵ, and written $\epsilon = (C_n, B_n, A_n)$, then all positive powers of ϵ appear in the expansion as $\epsilon^m = (C_{mn}, B_{mn}, A_{mn})$, and the reciprocal of ϵ^m namely ϵ^{-m} is the mth power of the reciprocal of ϵ. If ϵ is not a fundamental unit, only some of the units may appear as convergents. We will now explain just what we mean by a fundamental unit.

Let us consider the case first, when there is only one real root and for convenience let it be positive. The units are essentially of two types, positive and reciprocal units. The positive units are such that their successive powers give points, which approach asymptotically the asymptotic line of the surface $N(x, y, z) = 1$, that is, C_{mn}, B_{mn}, A_{mn}, form positive and increasing sets of numbers as m increases, such that C_{mn}/A_{mn} and B_{mn}/A_{mn} approach $q+p\theta+\theta^2$ and $p + \theta$ as limits. The smallest set of integers (C_n, B_n, A_n) which satisfy this condition defines the fundamental unit $\epsilon = C_n + B_n\theta + A_n\theta^2$. From the very nature of our work C_n, B_n, A_n are integers and hence the fundamental unit in the field $K(\theta)$* and the fundamental unit as defined here, will be the same only if the minimal basis is $1, \theta, \theta^2$, and may or may not be the same if the basis is not $1, \theta, \theta^2$.† The numbers we deal with here form a subset of the numbers in the field and hence it is desirable to call our fundamental unit, as defined here, the *fundamental unit in the chain.* The fundamental unit may or may nor appear as a convergent set in the expansion and in the table at the end of this paper, this will be indicated by calling the unit which is determined from the first period ϵ' (C_n, B_n, A_n) if $\epsilon' \neq \epsilon$, and indicating the relation between ϵ' and ϵ.

If the cubic has three real roots, we will consider a fundamental unit corresponding to each root. Let $\epsilon_1 = C_n + B_n\theta_1 + A_n\theta_1^2$ be a unit corresponding to one root θ_1 of the given cubic, and let $\epsilon_1^m = (C_{mn} + B_{mn}\theta_1 + A_{mn}\theta_1^2)$. If C_n, B_n, A_n are such that $N(C_n, B_n, A_n) = 1$, and if their absolute values are the smallest integers which can be found, such that some power of ϵ, leads to a normal expansion, so that

$$\lim_{m\to\infty} \frac{B_{mn}}{A_{mn}} = p + \theta_1, \text{ and } \lim_{m\to\infty} \frac{C_{mn}}{A_{mn}} = q + p\theta_1 + \theta_1^2,$$

* For a discussion of this subject, see for example L. W. Reid, *Tafel der Klassenanzahl für kubische Zahlkorper.* Dissertation, Göttingen (1899), pp. 25-28, or W. E. H. Berwick, *loc. cit.*

† The author has obtained periodic expansions corresponding to the case when the basis is not $1, \theta, \theta^2$, but they are not normal and will not be discussed here.

then ϵ_1, is defined as the *fundamental unit of the chain corresponding to* θ_1. Similarly ϵ_2 and ϵ_3 are defined as fundamental units corresponding to θ_2 and θ_3. These three units are not independent,* but are connected by a relation of the type

$$\epsilon_1^{m_1}\epsilon_2^{m_2}\epsilon_3^{m_3} = +1.$$

where m_1, m_2, m_3 are positive or negative integers. Even if the fundamental units, ϵ_1, ϵ_2, ϵ_3, are obtained from the corresponding convergent sets $(C_n,\ B_n,\ A_n)$, all the positive units are not given by convergent sets, because every number of the type,

$$\eta = \epsilon_i^m\,\epsilon_j^n,$$

is a unit and as it is not of a simple type it does not appear as a convergent set.

17. *Expansions with Given Periods.* It is possible to avoid some of the computation by determining the cubic, which has a given normal expansion. However, except in a few simple cases, the algebraic and numerical work involved is greater than the computation involved by other methods and not many of the resulting cubics are within the scope of the table given. We will consider two of the simple cases, which frequently occur, when $p = 0$, so that our cubic is of the type $x^3 + qx + r = 0$.

If the normal expansion of $(1,\ \theta,\ q + \theta^2)$ has only one partial quotient in the period, so that

$$(46)\qquad (1,\ \theta,\ q+\theta^2) = (p_1,\ q_1;\ p_2,\ p_1^2 + 2q_1 - q;\ |\ p_1 + p_2,\ p_1^2 + 2q_1 - q;),$$

then the purely periodic part is such that

$$(49)\qquad \bar{u}_3 : \bar{v}_3 : \bar{w}_3 = 1 : p_2 + \theta : q_1 + p_1\theta + \theta^2.$$

This comes from using the conditions (36) and (37), and (39) with $p = 0$.

If $1,\ p_2 + \theta,\ q_1 + p_1\theta + \theta^2$, be expanded into a continued fraction with the partial quotient set $(p_1 + p_2,\ q_2;)$, the next line of the expansion, giving $u,\ v,\ w$ is

$$-p_1 + \theta,\quad q - q_1 - p_1^2 + p_1\theta + \theta^2,\quad 1,$$

and these numbers are also in the ratio of $\bar{u}_3 : \bar{v}_3 : \bar{w}_3$. By equating the values of $\bar{v}_3/\bar{u}_3$ and $\bar{w}_3/\bar{u}_3$, we find

$$(48)\qquad -p_1^2 + q_1 + q + p_1\theta + \theta^2 = -p_1p_2 + (p_2 - p_1)\theta + \theta^2.$$

$$(49)\qquad 1 = -p_1q_1 - r.$$

From these equations, we find

* See for example, L. W. Reid, *loc. cit.*, p. 34.

(50) $p_2 = 2p_1;\quad q_1 = p_1^2 + q;\quad -r = p_1^3 + p_1 q + 1.$

In conclusion then, if θ is a root of $x^3 + qx + r = 0$, where

$$r = -(p_1^3 + p_1 q + 1),\quad p_1 \text{ and } q \text{ being arbitrary, then}$$

(46′) $(1, \theta, q + \theta^2) = (p_1, p_1^2 + q; 2p_1, 3p_1^2 + q; \mid 3p_1, 3p_1^2 + q;)$

and the fundamental unit in the chain is $(p_1^2 + q, p_1, 1)$. We will see later that this statement needs a slight modification. If, in particular, $p_1 = 0$, we have $p_1 = p_2 = p_3 = 0$; $q_1 = q_2 = q_3 = q$; $r = -1$, so that if θ is a root of $x^3 + qx - 1 = 0$, then

(46″) $$(1, \theta, q + \theta^2) = (0, q;)$$

If $p_1 = 1$, $r = -q - 2$, and

$$(1, \theta, q + \theta^2) = (1, 1 + q; 2, 3 + q; \mid 3, 3 + q;).$$

If $p_1 > 1$, the equations obtained are beyond the scope of the table computed.

When p_1 is negative, besides (46′), we must take into consideration, the following theorem due to Lehmer.*

"If the characteristic cubic has one root ρ_1, whose modulus is greater than the modulus of either of the other two, then the fractions B_n/A_n and C_n/A_n, approach, as $n \to \infty$, limits which are cubic irrationalities $\cdots$. If the characteristic cubic has two imaginary roots, whose common modulus is greater than the absolute value of the real root, then these fractions do not approach any limit."

In the case under consideration, the characteristic cubic $C(\rho) = 0$ is

(51) $$C(\rho) = \rho^3 - (3p_1^2 + q)\rho^2 - 3p_1\rho - 1 = 0.$$

When q is positive, it is readily seen that (51) has a real root greater than 1, since $C(1)$ is negative and $C(\infty)$ positive. Hence $C(\rho) = 0$ satisfies the first condition of the theorem, and C_n/A_n and B_n/A_n actually converge to $\theta^2 + q$ and θ as limits. When q is negative and p_1 positive and if $|q| < 3p_1^2 + 3p_1$, then $C(1)$ is negative and $C(\infty)$ positive, so that the first condition is satisfied. If $|q| > 3p_1^2 + 3p_1$, the discriminant of (51) is found to be positive, and hence (51) has three real roots and the first condition is again satisfied.

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* D. N. Lehmer, *Proceedings of the National Academy of Sciences*, Vol. 4 (1918), pp. 360-364.

The remaining case, q and p_1 both negative, sometimes leads to the first and sometimes to the second condition and under the second condition we do not have an expansion of cubic irrationalities. A numerical illustration will make this clear. Take $p_1 = -1$, so that equation under consideration is

$$x^3 + qx + q = 0, \qquad (q \text{ negative}). \tag{52}$$

The apparent expansion is

$$(1, \theta, q + \theta^2) = (-1, 1 + q; -2, 3 + q; \mid -3, 3 + q;), \tag{53}$$

and the fundamental unit is $(1 + q, -1, 1)$. The characteristic cubic is

$$C(\rho) = \rho^3 - (q + 3)\rho^2 + 3\rho - 1 = 0. \tag{54}$$

Its discriminant is

$$\Delta = q^2(-4q - 27).$$

It is readily seen that $C(0) = -1$; $C(1) = -q =$ positive number, and $C(\rho)$ has a root between 0 and 1. Also if $-q \leqq 6$, Δ is negative, while if $-q \geqq 7$, Δ is positive. In the former case $(-q \leqq 6)$, the second condition is satisfied and C_n/A_n and B_n/A_n do not approach limits and (53) does not represent an expansion in this case. It is worthy of note that $(1 + q, -1, 1)$ is the reciprocal of the fundamental unit, and hence the fundamental unit can be found. In the other case $(-q \geqq 7)$, the first condition of the theorem is satisfied, and it is found that (53) is the expansion corresponding to one of the negative roots of (52). In conclusion then, the statement of equation (46′) must be modified by adding the statement: providing A_n, B_n, C_n are actual convergents.

A second type of normal expansion which occurs frequently in the table is one with three sets of partial quotients in the period. We again consider the equation $x^3 + qx + r = 0$, such that the given expansion is

$$(1, \theta, q + \theta^2) = (p_1, q_1; p_2, q_2; \mid p_3, q_2; \ p_1 + p_2, q_4; p_1 + p_2, q_2;). \tag{55}$$

Let us consider the special case of $p_1 = 0$ so that $|\theta|$ is less than 1. Then the greatest integer in $q + \theta^2$ is q, and we select $q_1 = q$, and this also makes $q_4 = q$. The resulting expansion is periodic from the beginning. If we form the expansion, we have

$$\begin{array}{cccc}
1 & \theta & q + \theta^2 & 0, q; \\
\theta & \theta^2 & 1 & 0, q_2; \\
\theta^2 & 1 - q_2\theta & \theta & p_3, q_2; \\
1 - q_2\theta - p_3\theta^2 & \theta - q\theta^2 & \theta^2 & \cdots\cdot \\
\cdots & \cdots & \cdots &
\end{array}$$

From the relations $v_4/u_4 = \theta$ and $w_4/u_4 = (q + \theta^2)$, we readily find, by equating corresponding coefficients of like irrationalities, that

$$(56) \qquad p_3 = 0; \quad q = -rq_2 = mq_2 \qquad (m = -r).$$

In conclusion, if θ is a root of $x^3 + mq_2x - m = 0$, then

$$(57) \qquad (1, \theta, q + \theta^2) = (0, mq_2; 0, q_2; 0, q_2;),$$

provided A_n, B_n, C_n are actual convergents. It is possible to find other special cases for (55), but they do not occur frequently and it will not be done here.

18. *Negative Continued Fractions.* It was found convenient in certain cases to use what is called a negative continued fraction. This is defined by the relations

$$(58) \qquad u_{n+1} = p_nu_n - v_n; \; v_{n+1} = q_nu_n - w_n; \; w_{n+1} = u_n.$$

The p's and q's are selected so the $|\, u\text{'s}\,|$ form a decreasing sequence of numbers. If the numbers are positive, this makes p_n the integer next above the greatest integer in v_n/u_n.

The corresponding recursion formulas are

$$(59) \qquad \begin{aligned} A_n &= q_nA_{n-1} - p_nA_{n-2} + A_{n-3}, \\ B_n &= q_nB_{n-1} - p_nB_{n-2} + B_{n-3}, \\ C_n &= q_nC_{n-1} - p_nC_{n-2} + C_{n-3}, \end{aligned}$$

with the initial conditions

$$(60) \qquad \begin{vmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

The main theorems concerning normal expansions hold without modification and other results involve a change in sign, since the determinant of three successive convergent sets is now -1. For example, if θ is a root of $x^3 + qx - (p_1^3 + p_1q - 1) = 0$, then the negative expansion for $1, \theta, q + \theta^2$, is of the form (46′). No proofs of these facts will be given as they are readily derived by merely changing certain signs in the original proofs. It is also readily shown, that if a negative expansion for $(1, \theta, q + \theta^2)$ is known, then a positive expansion for $(1, -\theta, q + \theta^2)$ is obtained by changing the signs of the p's.

V. CONSTRUCTION AND EXPLANATION OF TABLE.

19. In making an extensive table of expansions, it may be pointed out that

$$(7) \qquad x^3 + px^2 + qx + r = 0,$$

need only be considered for values of $p = 0, 1, 2$.

Let θ be a root of (7), and let

$$(61) \qquad (1, p + \theta, q + p\theta + \theta^2) = (p_1, q_1; p_2, q_2; p_3, q_3; \cdots).$$

Decrease the roots of (7) by t, obtaining

$$(7') \qquad x^3 + p'x^2 + q'x + r' = 0,$$

where

$$p' = p + 3t, \quad q' = q + 2pt + 3t^2.$$

Let θ' be the corresponding root of (7′) and let

$$(61') \qquad (1, p' + \theta, q' + p'\theta' + \theta'^2) = (p'_1, q'_1; p'_2, q'_2; \cdots).$$

Then by actually constructing the expansion, it is found $p'_1 = p_1 + 2t$; and if we select $q'_1 = t^2 + p_1 t + q_1$, then $p'_2 = p_2 + t$, and the balance of the expansion (61′) is identical with (61). Hence we need consider the values of p in (7) which are less than 3, namely 0, 1, 2, although -1, 0, 1 would do just as well.

20. *Table for* $x^3 + qx + r = 0$. Because of the information already available in the paper by Reid, which we have previously mentioned, the author considered the case $p = 0$. The table at the end of this paper gives the units and the normal expansions corresponding to $x^3 + qx + r = 0$ for q between -9 and $+9$, and $-r$ between 1 and 9. The order and numbering of Reid's table is retained for reference. However, in this table the sign of the root is changed in order to make it positive, when there is only one real root. If there are three roots, they are called θ_1, θ_2, θ_3 in order of magnitude, θ_1 being the largest. The fundamental unit of the chain is designated by $\epsilon\,(x, y, z)$, and the relation between the unit which is determined by the first period $\epsilon'(C_n, B_n, A_n)$ and ϵ is given. If there are three real roots, the relation between ϵ_1, ϵ_2, ϵ_3, is also given. The period of the expansion is separated from the non-recurring sets by a vertical bar |, and a complete period is given or indicated. For example the entry corresponding to the numerical example given before $x^3 + x - 5 = 0$, would be

34. $x^3 + x - 5 = 0 \qquad (13, 6, 4) \quad \epsilon' = \epsilon^2$

$(1, 3; 0, 1; \mid 1, 1; 1, 1; 3, 4; 3, 1; 1, 1; 1, 1; 1, 6; 1, 1;)$.

In case the period is long, only part of the expansion is written, and the balance can be supplied from the palindromic relations and the conditions for a normal expansion (38, 39). The middle elements of the skew-palindromic part are enclosed in parentheses. If n is even, two p's and one q will be in parentheses, and if n is odd one p and two q's. For example the entry just written could also be written

$$(1, 3; 0, 1; \mid 1, 1; 1, 1; (3), (4); (3), \cdots).$$

If the expansion is purely periodic no vertical bar is used. A negative expansion is indicated by NE before the expansion, which means that formulas (58), (59), (60) must be used.

21. *Computation.* There are several methods of determining the expansion corresponding to a given equation. If the equation is not of a form, whose expansion is of a known type, and if a unit is known, we attempt to expand the unit into a finite fraction, which after p_1, q_1; is skew-palindromic. We do not usually use the fundamental unit, but some power of it, so that x, y, z are large enough to make x/z and y/z fair approximations to $q + \theta^2$ and θ. We also do not use the fundamental unit, because at times it does not occur in the expansion. The expansion can then be checked for the conditions of a normal expansion, by considering higher powers of the unit, or forming the rationalized complete quotient sets or other devices. The rationalization process is long and is avoided when possible.

As a numerical example, consider $x^3 - 3x - 6 = 0$. It is found that $\epsilon = (79, 73, 31)$, $\epsilon^2 = (33397, 30878, 13110)$. The expansion is below

				A	B	C	
				1	0	0	
				0	1	0	
u	v	w	$p, q;$	0	0	1	
13110	30878	33397	2, 2;	1	2	2	
4658	7177	13110	1, 2;	2	5	5	
2519	3794	4658	1, 1;	3	7	8	
1275	2139	2519	1, 1;	6	14	15	
864	1244	1275	1, 1;	11	26	28	
380	411	864	1, 2;	31	73	79	ϵ
31	104	380	3, 11;	380	. . .		
11	39	31	3, 2;	864			
6	9	11	1, 1;	1275			
3	5	6	1, 1;	2519			
2	3	3	1, 1;	4658			
1	1	2	1, 2;	13110			
0	0	1					

The only place we did not take q as the greatest integer in w/u was for $q_7 = 11$ and this is done to make the expansion normal.

If a unit is not known, we take as $u_1 : v_1 : w_1$, numbers which are approximations to $1 : \theta : q + \theta^2$, and expand them, trying for a normal expansion, and computing the values of m as we proceed. Thus for $x^3 + 5x - 2 = 0$, we have $\theta \doteq .3883$; $\theta^2 \doteq .1508$. The expansion is below

				1	0	0	
				0	1	0	
				0	0	1	m
10000	3883	51508	0, 5;	1	0	5	4
3883	1508	10000	0, 2;	2	1	10	4
1508	2234	3883	1, 2;	5	2	26	2
726	867	1508	1, 2;	13	5	67	1
141	56	726	0, 5;		. . .		
56	21	141	0, 2;				
21	29	56	1, 2;				
8	14	21	1, 2;				
. . .	. . .						

We readily recognize that the expansion is normal and purely periodic. This may be checked by considering the expansion for some power of ϵ.

If the expansion is not obtained by this last method, we usually obtain values, which satisfy $N(x, y, z) = m$, for small values of m. By combining them according to various methods, we are able to find a unit and check to see whether it is a power of a unit.* Having a unit we expand it by the first method. In many cases the numerical calculations are long and tedious, and in the few cases where a blank appears in the table, the author feels that there is a normal expansion but that the effort necessary to find it, is probably not worth while.

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* For a discussion of these methods with reference to $x^3 - D = 0$, see Meissel, E., *Beitrag zur Pell'schen Gleichung höherer Grade*, or Wolfe, C., "On the Indeterminate Cubic Equation," *University of California Publications in Mathematics*, Vol. 1 (1923), pp. 359-369.

Table of Units and Normal Expansions.

	$\epsilon(x, y, z)$		*Expansion.*
1. $x^3-1=0$	reducible.		
2. $x^3+x-1=0$	$(1,0,1)$		$(0,1;)$
3. $x^3-x-1=0$	$(0,1,1)$		$(1,0;2,2;\mid 3,2;)$
4. $x^3-2=0$	$(1,1,1)$		$(1,1;2,3;\mid 3,3;)$
5. $x^3+x-2=0$	reducible.		
6. $x^3-x-2=0$	$(1,1,1)$	$\epsilon'=\epsilon_1^4$	$(1,1;0,1;\mid 1,1;1,1;3,2;3,1;1,1;1,1;1,4;1,1;)$
7. $x^3+2x-2=0$	$(3,1,1)$		$(0,2;0,1;0,1;)$
8. $x^3-2x-2=0$	$(1,1,0)$	$\epsilon'=\epsilon^2$	$(1,1;0,1;\mid 1,5;1,1;)$
9. $x^3+2x-1=0$	$(2,0,1)$		$(0,2;)$
10. $x^3-2x-1=0$	reducible.		
11. $x^3-3=0$	$(4,3,2)$		$(1,2;0,2;\mid 1,5;1,2;)$
12. $x^3+x-3=0$	$(2,1,1)$		$(1,2;2,4;\mid 3,4;)$
13. $x^3-x-3=0$	$(2,2,1)$		$(1,1;1,1;\mid 2,4;2,1;)$
14. $x^3+2x-3=0$	reducible.		
15. $x^3-2x-3=0$	$(2,2,1)$	$\epsilon'=\epsilon^2$	$(1,1;0,1;\mid 0,1;2,3;2,1;0,1;1,5;1,1;)$
		or $\epsilon'=\epsilon$	*NE* $(2,2;4,10;\mid 6,10;)$
16. $x^3+3x-3=0$	$(4,1,1)$		$(0,3;0,1;0,1;)$
17. $x^3-3x-3=0$	$(1,1,0)$	$\epsilon'=\epsilon^2$	$(2,1;4,9;\mid 6,9;)$
18. $x^3+3x-1=0$	$(3,0,1)$		$(0,3;)$
19. $x^3-3x-1=0$	$\epsilon_1(1,1,0)$	$\epsilon_1'=\epsilon_1^4$	$(1,0;0,1;\mid 0,1;2,2;2,1;0,1;1,4;1,1;)$
		or $\epsilon_1'=\epsilon_1^2$	*NE* $(2,1;4,9;\mid 6,9;)$
	$\epsilon_2(-3,0,1)$		$(0,-3;)$
$\epsilon_1^2\epsilon_2^2\epsilon_3=1.$	$\epsilon_3(-1,-3,2)$		$(-1,-1;0,-1;\mid 1,-1;-1,2;-1,-1;)$
20. $x^3+3x-2=0$	$(17,3,5)$	$\epsilon'=\epsilon^2$	$(0,2;2,1;\mid 2,2;3,1;2,1;3,2;2,1;2,1;2,1;)$
21. $x^3-3x-2=0$	reducible.		
22. $x^3-4=0$	$(5,3,2)$		$(1,1;2,1;\mid 1,1;3,3;3,1;)$
23. $x^3+x-4=0$	$(61,29,21)$		$(1,2;2,2;\mid 1,1;2,1;1,2;3,4;3,2;)$
24. $x^3-x-4=0$	$(11,9,5)$	$\epsilon'=\epsilon^2$	$(1,2;0,1;\mid 0,3;0,1;0,1;3,2;3,1;0,1;0,3;0,1;1,6;1,1;)$
25. $x^3+2x-4=0$	$(3,1,1)$		$(1,3;2,5;\mid 3,5;)$
26. $x^3-2x-4=0$	reducible.		
27. $x^3+3x-4=0$	reducible.		
28. $x^3-3x-4=0$	$(9,11,5)$		$(2,2;-1,5;\mid 1,11;1,5;)$
29. $x^3+4x-4=0$	$(5,1,1)$		$(0,4;0,1;0,1;)$
30. $x^3-4x-4=0$	$(1,1,0)$	$\epsilon'=\epsilon^3$	$(2,1;1,1;\mid 2,1;3,10;3,1;)$

31. $x^3+4x-1=0$	$(4, 0, 1)$		$(0, 4;)$
32. $x^3-4x-1=0$	$\epsilon_1(0, 2, 1)$		$(2, 0; 4, 8; \mid 6, 8;)$
	$\epsilon_2(-4, 0, 1)$		$(0, -4;)$
$\epsilon_1\epsilon_2\epsilon_3=1.$	$\epsilon_3(0, -2, 1)$		$(-2, 0; -4, 8; \mid -6, 8;)$
33. $x^3+4x-2=0$	$(17, 2, 4)$		$(0, 4; 0, 2; 0, 2;)$
34. $x^3-4x-2=0$	$\epsilon_1(17, 42, 19)$		$(2, 0; 4, 4; \mid 3, 4; 6, 8; 6, 4;)$
	$\epsilon_2(-15, -2, 4)$		$(0, -4; 0, -2; 0, -2;)$
$\epsilon_1\epsilon_2^2\epsilon_3^3=1$	$\epsilon_3(1, 2, -1)$		$(-1, -1; 0, -1; \mid -1, 3; -1, -1;)$
35. $x^3+4x-3=0$	$(40, 6, 9)$		$(0, 4; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1;)$
36. $x^3-4x-3=0$	reducible.		
37. $x^3-5=0$	$(41, 24, 14)$		$(1, 2; 1, 1; \mid 1, 2; 3, 2; 1, 1; 2, 5; 2, 1;)$
38. $x^3+x-5=0$	$(13, 6, 4)$	$\epsilon'=\epsilon^2$	$(1, 3; 0, 1; \mid 1, 1; 1, 1; 3, 4; 3, 1; 1, 1; 1, 1; 1, 6; 1, 1;)$
		or $\epsilon'=\epsilon$	$NE(2, 4; 2, 2; \mid 0, 2; 4, 11; 4, 2;)$
39. $x^3-x-5=0$	$(3, 2, 1)$	$\epsilon'=\epsilon^2$	$(1, 2; 0, 1; \mid 0, 1; 2, 4; 2, 1; 0, 1; 1, 6; 1, 1;)$
		or $\epsilon'=\epsilon$	$NE(2, 3; 4, 11; \mid 6, 11;)$
40. $x^3+2x-5=0$	$(34, 12, 9)$	$\epsilon'=\epsilon^2$	$(1, 3; 2, 3; \mid 0, 3; 0, 7; 0, 3; 0, 3; 3, 5; 3, 3;)$
41. $x^3-2x-5=0$	$(2, 2, 1)$		$(2, 2; 4, 10; \mid 6, 10;)$
42. $x^3+3x-5=0$	$(4, 1, 1)$		$(1, 4; 2, 6; \mid 3, 6;)$
43. $x^3-3x-5=0$	$(2, 2, 1)$	$\epsilon'=\epsilon^2$	$(2, 1; 4, 3; \mid 2, 3; 6, 9; 6, 3;)$
44. $x^3+4x-5=0$	reducible.		
45. $x^3-4x-5=0$	$(4, 5, 2)$		$(2, 2; 0, 2; \mid 2, 12; 2, 2;)$
46. $x^3+5x-5=0$	$(6, 1, 1)$		$(0, 5; 0, 1; 0, 1;)$
47. $x^3-5x-5=0$	$(1, 1, 0)$	$\epsilon'=\epsilon^3$	$(2, 1; 1, 2; \mid -1, 2; 3, 11; 3, 2;)$
		or $\epsilon'=\epsilon^5$	$(2, 1; 1, 1; \mid 1, 2; 0, 1; 2, 1; 0, 2; 1, 1; 3, 11; 3, 1;)$
48. $x^3+5x-1=0$	$(5, 0, 1)$		$(0, 5;)$
49. $x^3-5x-1=0$	$\epsilon_1(4, 21, 9)$		$(2, 0; 1, 3; \mid 0, 3; 3, 9; 3, 3;)$
	$\epsilon_2(-5, 0, 1)$		$(0, -5;)$
$\epsilon_1\epsilon_2^2\epsilon_3^2=1.$	$\epsilon_3(-1, -2, 1)$		$NE(-2, -1; -4, 7; \mid -6, 7;)$
50. $x^3+5x-2=0$	$(67, 5, 13)$		$(0, 5; 0, 2; 1, 2; 1, 2;)$
51. $x^3-5x-2=0$	reducible.		
52. $x^3+5x-3=0$	$(85, 9, 16)$		$(0, 4; \mid 2, 1; 2, 1; 5, 1; 2, 1; 2, 3;)$
53. $x^3-5x-3=0$	$\epsilon_1(5, 10, 4)$	$\epsilon_1'=\epsilon_1^2$	$(2, 1; 0, 2; \mid 0, 2; 4, 9; 4, 2; 0, 2; 2, 11; 2, 2;)$
	$\epsilon_2(-4, -1, 1)$		$NE(-1, -4; -2, -2; \mid -3, -2;)$
$\epsilon_1\epsilon_2^2\epsilon_3=1.$	$\epsilon_3(-1, -2, 1)$		$(-2, -1; -4, 7; \mid -6, 7;)$
54. $x^3+5x-4=0$	$(801, 105, 145)$		$(0, 5; 0, 1; 0, 1; 0, 1; 1, 1; 1, 1; 5, 1; 1, 1; 1, 1; 0, 1; 0, 1;)$
55. $x^3-5x-4=0$	reducible.		

56. $x^3-6=0$ (109, 60, 33) (1, 3; 0, 1; | 0, 1; 2, 1; 0, 1; 0, 1; 0, 1; 2, 1; 0, 1; 1, 7; 1, 1;)

57. $x^3+x-6=0$ (11, 5, 3) (1, 2; 2, 1; | 0, 1; 0, 1; 0, 1; 3, 4; 3, 1;)
or *NE*(2, 4; 1, 3; | 3, 11; 3, 3;)

58. $x^3-x-6=0$ reducible.

59. $x^3+2x-6=0$ (1129, 399, 274) (1, 4; 0, 2; | 0, 2; 2, 1; 0, 1; 1, 1; 0, 1; 2, 2; 0, 2; 1, 11; 1, 2;)

60. $x^3-2x-6=0$ (77, 61, 28) (2, 2; 4, 5; | 3, 5; 6, 10; 6, 5;)

61. $x^3+3x-6=0$ (11521, 3185, 2473) (1, 4; 2, 3; | 1, 1; 3, 1; 0, 1; 0, 1; (1), (2); (1), · · ·)

62. $x^3-3x-6=0$ (79, 73, 31) (2, 2; 1, 2; | 1, 1; 1, 1; 1, 1; 1, 2; 3, 11; 3, 2;)

63. $x^3+4x-6=0$ (5, 1, 1) (1, 5; 2, 7; | 3, 7;)

64. $x^3-4x-6=0$ (95, 101, 40) (2, 2; 0, 1; | 1, 1; 1, 1; 5, 1; 1, 1; 1, 1; 2, 12; 2, 1;)

65. $x^3+5x-6=0$ reducible.

66. $x^3-5x-6=0$ (29, 35, 13) (2, 2; 0, 1; | 1, 2; 2, 2; 1, 1; 2, 13; 2, 1;)

67. $x^3+6x-6=0$ (7, 1, 1) (0, 6; 0, 1; 0, 1;)
or *NE*(1, 7; 2, 9; | 3, 9;)

68. $x^3-6x-6=0$ (1, 1, 0) $\epsilon'=\epsilon^5$ (2, 1; 1, 1; | 0, 1; 3, 1; 0, 1; 0, 1; 0, 1; 3, 1; 0, 1; 3, 12; 3, 1;)

69. $x^3+6x-1=0$ (6, 0, 1) (0, 6;)

70. $x^3-6x-1=0$ ϵ_1(2974, 19020, 7521) (2, 0; 0, 1; | 1, 1; 1, 4; 1, 1; 0, 1; 1, 1; (3), (2); (3), · · ·)
ϵ_2(−6, 0, 1) (0, −6;)
$\epsilon_1\epsilon_2^9\epsilon_3^2=1.$ ϵ_3(−14, −78, 33) (−2, 0; 1, −2; | −4, 4; −4, −2; −1, 10; −1, −2;)

71. $x^3+6x-2=0$ (55, 3, 9) (0, 6; 0, 3; 0, 3;)

72. $x^3-6x-2=0$ ϵ_1(13, 44, 17) (2, 0; 1, 1; | 2, 1; 6, 1; 2, 1; 3, 10; 3, 1;)
ϵ_2(−53, −3, 9) (0, −6; 0, −3; 0, −3;)
$\epsilon_1\epsilon_2^2\epsilon_3=1.$ ϵ_3(−7, −18, 8) (−2, −1; 0, −2; | 4, −2; −2, 8; −2, −2;)

73. $x^3+6x-3=0$ (25, 2, 4) (0, 6; 0, 2; 0, 2;)

74. $x^3-6x-3=0$ ϵ_1(10, 24, 9) (2, 0; 1, 1; | 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 3, 10; 3, 1;)
ϵ_2(−23, −2, 4) (0, −6; 0, −2; 0, −2;)
$\epsilon_1\epsilon_2\epsilon_3^2=1.$ ϵ_3(−2, −2, 1) *NE*(−2, −2; −4, 6; | −6, 6;)

75. $x^3+6x-4=0$ (51, 5, 8) (0, 5; | 2, 1; 2, 3; 2, 1; 2, 4;)

76. $x^3-6x-4=0$ reducible.

77. $x^3+6x-5=0$ (8117, 938, 1234) (0, 6; 0, 1; 0, 1; 0, 3; −1, 1; 1, 4; 1, 2; 1, 4; 1, 1; −1, 3; 0, 1; 0, 1;)

78. $x^3-6x-5=0$ reducible.

79. $x^3-7=0$ (4, 2, 1) $\epsilon'=\epsilon^2$ (1, 3; 0, 1; | 0, 1; 2, 5; 2, 1; 0, 1; 1, 7; 1, 1;)
or $\epsilon'=\epsilon$ *NE*(2, 4; 4, 12; | 6, 12;)

80. $x^3+x-7=0$ (16, 7, 4) $\epsilon'=\epsilon^2$ (1, 3; 1, 1; | 0, 2; 0, 1; 0, 1; 3, 4; 3, 1; 0, 1; 0, 2; 0, 1; 2, 6; 2, 1;)
or $\epsilon'=\epsilon$ NE(2, 4; 0, 4; | 2, 11; 2, 4;)

81. $x^3-x-7=0$ (3, 2, 1) (2, 3; 4, 11; | 6, 11;)

82. $x^3+2x-7=0$ (290, 102, 95) (1, 4; 0, 1; | 0, 1; 0, 1; 0, 3; 3, 3; 0, 1; 0, 1; 0, 1; 1, 7; 1, 1;)

83. $x^3-2x-7=0$ (372, 271, 120) $\epsilon'=\epsilon^2$ (2, 3; 0, 3; | 2, 2; 2, 3; 3, 3; (6), (10); (6), · · ·)

84. $x^3+3x-7=0$ (25, 7, 5) (1, 4; 2, 2; | 1, 2; 3, 6; 3, 2;)

85. $x^3-3x-7=0$ (6, 5, 2) (2, 2; 2, 2; | 4, 11; 4, 2;)

86. $x^3+4x-7=0$ (1026, 231, 184) $\epsilon'=\epsilon^2$
(1, 4; 6, 3; | 5, 5; 2, 2; 2, 3; —5, 11; —5, 3; 2, 2; 2, 5; 5, 3; 7, 5; 7, 3;)

87. $x^3-4x-7=0$ (46, 44, 17) $\epsilon'=\epsilon_2$ (2, 2; 1, 1; | 3, 3; 3, 1; 1, 1; (6), (8); (6), · · ·)
or $\epsilon'=\epsilon$ (2, 3; —1, 1; | 1, 1; 2, 1; 2, 1; 1, 1; 1, 4; 1, 1;)

88. $x^3+5x-7=0$ (6, 1, 1) (1, 6; 2, 8; | 3, 8;)

89. $x^3-5x-7=0$ (3, 3, 1) (2, 1; 2, 1; | 4, 11; 4, 1;)

90. $x^3+6x-7=0$ reducible.

91. $x^3-6x-7=0$ (5, 6, 2) (2, 2; 0, 2; | —2, 2; 2, 14; 2, 2;)

92. $x^3+7x-7=0$ (8, 1, 1) (0, 7; 0, 1; 0, 1;)

93. $x^3-7x-7=0$ ϵ_1(2, 3, 1) (3, 2; 6, 20; | 9, 20;)
ϵ_2(—6, —1, 1) (—1, —6; —2, —4; | —3, —4;)
$\epsilon_1\epsilon_2\epsilon_3=1.$ ϵ_3(—3, —2, 1) (—2, —3; —4, —5; | —6, 5;)

94. $x^3+7x-1=0$ (7, 0, 1) (0, 7;)

95. $x^3-7x-1=0$ ϵ_1(18, 133, 49) (2, 0; 0, 1; | 0, 1; 0, 1; 1, 4; 1, 1; 1, 1; 0, 1; 2, 11; 2, 1;)
ϵ_2(—7, 0, 1) (0, —7;)
$\epsilon_1\epsilon_2{}^5\epsilon_3{}^2=1.$ ϵ_3(—2, —13, 5) (—2, 0; 1, —3; | —4, —3; —1, 11; —1, —3;)

96. $x^3+7x-2=0$ (137865, 5501, 19473)
(0, 5; 7, 3; | 1, 2; 1, 1; 1, 1; 0, 1; 0, 1; 0, (1); (9), (1); · · ·)

97. $x^3-7x-2=0$ ϵ_1(13, 50, 18) (2, 0; 0, 1; | 0, 1; 0, 3; 0, 3; 0, 1; 0, 1; 2, 11; 2, 1;)
ϵ_2(97, 4, —14) (0, —8; | —3, —3; 1, —1; 1, —3; —3, —9;)
$\epsilon_1\epsilon_2\epsilon_3=1.$ ϵ_3(—3, —10, 4) $\epsilon_3{}'=\epsilon_3{}^2$
(—2, —1; 0, —2; | 0, —2; —4, 7; —4, —2; 0, —2; —2, 9; —2, —2;)

98. $x^3+7x-3=0$ (86, 5, 12) (0, 7; 0, 2; 0, 1; 1, 1; 0, 2;)

99. $x^3-7x-3=0$ ϵ_1(1, 3, 1) (2, 1; 0, 1; | 2, 13; 2, 1;)
ϵ_2(14, 1, —2) (0, —6; | 2, —2; 2, —5;)
$\epsilon_1\epsilon_2{}^2\epsilon_3=1$ ϵ_3(25, 48, —20) (—2, —1; 1, —3; | —1, —3; —1, —3; —1, 9; —1, —3;)

100. $x^3+7x-4=0$ (42115, 3161, 5769) (0, 6; | 2, 1; 2, 2; 3, 4; 3, 2; 7, 2; 3, 4; 3, 2; 2, 1; 2, 5;)

101. $x^3-7x-4=0$ $\epsilon_1(105, 220, 76)$ (2, 1; 0, 1; | 0, 2; 1, 3; 2, 3; 1, 2; 0, 1; 2, 13; 2, 1;)

$\epsilon_2(33, 3, -5)$ $\epsilon'_2=\epsilon_2^2$ (0, —7; | —1, —2; 1, 3; 3, —8; 3, 3; 1, —2; —1, 7;)

$\epsilon_1\epsilon_2^2\epsilon_3^3=1$ $\epsilon_3(5, 7, -3)$ (—2, —2; —1, —3; | —3, 7; —3, —3;)

102. $x^3+7x-5=0$ (67, 6, 9) (0, 7; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1;)

103. $x^3-7x-5=0$ $\epsilon_1(2, 3, 1)$ (2, 0; 1, 1; | 0, 1; 3, 11; 3, 1;)

or *NE*(3, 2; 6, 20; | 9, 20;)

$\epsilon_2(-6, -1, 1)$ *NE*(—1, —6; —2, —4; | —3, —4;)

$\epsilon_1\epsilon_2\epsilon_3=1$ $\epsilon_3(-3, -2, 1)$ *NE*(—2, —3; —4, 5; | —6, 5;)

104. $x^3+7x-6=0$ (2719205, 280985, 356851) (0, 7; 0, 1; 0, 1; 0, 2; 1, 1; 1, 3; 4, 1; 0, 1; 0, 1; 1, 2; (3), (5); (3), · · ·)

105. $x^3-7x-6=0$ reducible.

106. $x^3-8=0$ reducible.

107. $x^3+x-8=0$ (157, 66, 36) (1, 3; 1, 1; | 0, 1; 1, 3; 0, 1; 0, 3; 1, 1; 0, 1; 2, 6; 2, 1;)

108. $x^3-x-8=0$ (133, 78, 36) $\epsilon'=\epsilon^2$ (2, 3; 4, 6; | 0, 6; 0, 13; 0, 6; 6, 11; 6, 6;)

109. $x^3+2x-8=0$ (43, 15, 9) (1, 4; 1, 1; | 2, 4; 2, 1; 2, 7; 2, 1;)

110. $x^3-2x-8=0$ (31, 21, 9) (2, 3; 1, 3; | 0, 3; 3, 12; 3, 3;)

111. $x^3+3x-8=0$ (21, 6, 4) $\epsilon'=\epsilon^2$ (1, 5; 0, 1; | 1, 1; 1, 1; (3), (6); (3), · · ·)

112. $x^3-3x-8=0$ (13, 10, 4) $\epsilon'=\epsilon^2$ (2, 3; 0, 2; | 0, 2; 4, 11; 4, 2; 0, 2; 2, 13; 2, 2;)

113. $x^3+4x-8=0$ (129, 30, 22) (1, 5; 2, 3; | —1, 3; —1, 3; 3, 7; 3, 3;)

or (1, 2; 10, 1; | 2, 1; 0, 1; 0, 1; 0, 1; 2, 1; 11, 1; 11, 1;)

114. $x^3-4x-8=0$ (9, 8, 3) (2, 2; 1, 1; | 0, 1; 0, 1; 0, 1; 3, 12; 3, 1;)

115. $x^3+5x-8=0$ (30949, 5842, 4754) $\epsilon'=\epsilon^2$

(1, 6; 2, 4; | 1, 3; 2, 1; 2, 3; 1, 1; 0, 1; 0, 1; 3, 2; 1, 4; (3), (8), (3), · · ·)

116. $x^3-5x-8=0$ (3, 3, 1) (2, 2; 1, 1; | 3, 13; 3, 1;)

117. $x^3+6x-8=0$ (7, 1, 1) (1, 7; 2, 9; | 3, 9;)

118. $x^3-6x-8=0$ (3, 3, 1) (2, 1; 1, 1; | 0, 1; 3, 12; 3, 1;)

119. $x^3+7x-8=0$ reducible.

120. $x^3-7x-8=0$ (5, 6, 2) $\epsilon'=\epsilon^2$ (3, 2; 6, 8; | 20, 8; 9, 20; 9, 8;)

121. $x^3+8x-8=0$ (9, 1, 1) (0, 8; 0, 1; 0, 1;)

122. $x^3-8x-8=0$ reducible.

<table>
<tr><td>123. $x^3+8x-1=0$</td><td>$(8, 0, 1)$</td><td>$(0, 8;)$</td></tr>
<tr><td>124. $x^3-8x-1=0$</td><td>$\epsilon_1(81, 676, 234)$</td><td>$(2, 0; 0, 1; \mid 0, 2; 1, 2; 1, 1; 0, 2;$
$0, 1; 1, 2; 1, 2; 0, 1; 2, 12; 2, 1;)$</td></tr>
<tr><td></td><td>$\epsilon_2(-8, 0, 1)$</td><td>$(0, -8;)$</td></tr>
<tr><td>$\epsilon_1\epsilon_2^4\epsilon_3=1.$</td><td>$\epsilon_3(-26, -199, 72)$</td><td>$(-3, 0; -2, 4; \mid 0, 2; 0, 2; 0, 4; -5,$
$17; -5, 4)$</td></tr>
<tr><td>125. $x^3+8x-2=0$</td><td>$(129, 4, 16)$</td><td>$(0, 8; 0, 4; 0, 4;)$</td></tr>
<tr><td>126. $x^3-8x-2=0$</td><td>$\epsilon_1(1, 3, 1)$ $\epsilon_1'=\epsilon_1^2$</td><td>$(2, 0; 0, 1; \mid 0, 1; 4, 8; 4, 1; 0, 1;$
$2, 12; 2, 1;)$</td></tr>
<tr><td></td><td>$\epsilon_2(-127, -4, 16)$</td><td>$(0, -8; 0, -4; 0, -4;)$</td></tr>
<tr><td>$\epsilon_1^3\epsilon_2^2\epsilon_3=1.$</td><td>$\epsilon_3(-951, -3451, 1281)$</td><td></td></tr>
<tr><td></td><td></td><td>$(-3, -1; 0, 2; \mid 1, 1; 0, 1; 2, 1; 2, 1;$
$0, 1; 0, 1; 0, 1; 2, 1; 2, 1; 0, 1;$
$1, 2; -3, 15; -3, 2;)$</td></tr>
<tr><td>127. $x^3+8x-3=0$</td><td>$(2764324, 125283, 339766)$</td><td></td></tr>
<tr><td></td><td></td><td>$NE(0, 8; 0, -2; -1, -2; 0, 1; 0, -1;$
$-3, -10; 2, (5); (-5), (5); \ldots)$</td></tr>
<tr><td>128. $x^3-8x-3=0$</td><td>reducible.</td><td></td></tr>
<tr><td>129. $x^3+8x-4=0$</td><td>$(33, 2, 4)$</td><td>$(0, 8; 0, 2; 0, 2;)$</td></tr>
<tr><td>130. $x^3-8x-4=0$</td><td>$\epsilon_1(1, 3, 1)$</td><td>$(3, 1; 6, 19; \mid 9, 19;)$</td></tr>
<tr><td></td><td>$\epsilon_2(31, -2, 4)$</td><td>$(0, -8; 0, -2; 0, -2;)$</td></tr>
<tr><td>$\epsilon_1\epsilon_2\epsilon_3=1.$</td><td>$\epsilon_3(-3, -5, 2)$</td><td>$(-2, -2; 0, -1; \mid 1, -1; -2, 8;$
$-2, -1;)$</td></tr>
<tr><td>131. $x^3+8x-5=0$</td><td>$(4764, 341, 570)$ $\epsilon'=\epsilon^2$</td><td></td></tr>
<tr><td></td><td></td><td>$(0, 7; 2, 1; \mid 2, 3; 1, 1; 2, 2; 0, 2; 1, 1;$
$2, 5; 2, (2); (3), (2); \cdots)$</td></tr>
<tr><td>132. $x^3-8x-5=0$</td><td>$\epsilon_1(776874, 1493568, 481729)$</td><td></td></tr>
<tr><td></td><td></td><td>$(3, 2; -4, 9; \mid 6, 6; 1, 1; 1, 2; 2, 9;$
$2, 1; 1, 1; 1, 9; 9, 9; (9), (19);$
$(9), \cdots)$</td></tr>
<tr><td></td><td>$\epsilon_2(-68, -6, 9)$</td><td>$(0, -8; \mid -1, -2; 2, 4; 2, -2; -1, -8;)$</td></tr>
<tr><td></td><td>$\epsilon_3(4, 5, -2)$</td><td>$(-2, -2; 0, -2; \mid -2, 8; -2, -2;)$</td></tr>
<tr><td>133. $x^3+8x-6=0$</td><td>$(289, 24, 34)$ $\epsilon'=\epsilon^2$</td><td>$(0, 7; \mid 2, 1; 3, 7; 2, 2; 0, 1; 0, 1;$
$(4), (2); (4), \cdots)$</td></tr>
<tr><td>134. $x^3-8x-6=0$</td><td>$\epsilon_1(427, 705, 224)$</td><td>$(4, 2; 0, -2; \mid 7, 9; 0, 2; 0, 9; 7, -2;$
$4, 28; 4, -2;)$</td></tr>
<tr><td></td><td>$\epsilon_2(-7, -1, 1)$</td><td>$NE(-1, -7; -2, -5; \mid -3, -5;)$</td></tr>
<tr><td>$\epsilon_1\epsilon_3^3\epsilon_3=1$</td><td>$\epsilon_3(-19, -16, 7)$</td><td>$(-2, -4; -4, -2; \mid 3, -2; -6, 4;$
$-6, -2;)$</td></tr>
<tr><td>135. $x^3+8x-7=0$</td><td>$(1177, 110, 136)$</td><td>$(0, 8; 0, 1; 0, 1; 0, 3; 0, 1; 0, 2; -1, 2;$
$0, 1; 0, 3; 0, 1; 0, 1;)$</td></tr>
<tr><td>136. $x^3-8x-7=0$</td><td>reducible.</td><td></td></tr>
<tr><td>137. $x^3-9=0$</td><td>$(4, 2, 1)$</td><td>$(2, 4; 4, 12; \mid 6, 12;)$</td></tr>
</table>

138. $x^3+x-9=0$	$(5,2,1)$	$\epsilon'=\epsilon^2$	$(1, 4; 0, 1; \mid 0, 1; 2, 6; 2, 1; 0, 1; 1, 8; 1, 1;)$
		$\epsilon'=\epsilon$	$NE(2, 5; 4, 13; \mid 6, 13;)$
139. $x^3-x-9=0$	$(16,9,4)$		$(2, 4; 0, 4; \mid 2, 13; 2, 4;)$
140. $x^3+2x-9=0$	$(817,282,160)$		$(1, 4; 1, 1; \mid 0, 2; 0, 1; 1, 2; 2, 1; 1, 1; 0, 1; 0, 2; 0, 1; 0, 1; (3), (5); (3), \cdots)$
141. $x^3-2x-9=0$	$(9550,6104,2545)$		$(2, 3; 1, 2; \mid 0, 1; 0, 1; 3, 4; 1, 2; 1, 2; 1, 4; 3, 1; 0, 1; 0, 1; 3, 15; 3, 2;)$
142. $x^3+3x-9=0$	$(28,8,5)$		$(1, 6; -1, 1; \mid 1, 2; 1, 1; 0, 10; 0, 1;)$
143. $x^3-3x-9=0$	$(7,5,2)$		$(2, 3; 0, 1; \mid 0, 1; 0, 1; 2, 13; 2, 1;)$
144. $x^3+4x-9=0$	$(172,41,28)$		$(1, 6; 0, 2; \mid 0, 2; 2, 2; 0, 2; 1, 9; 1, 2;)$
145. $x^3-4x-9=0$	$(113,92,34)$		$(2, 3; 0, 1; \mid 0, 1; 1, 1; 0, 3; 0, 1; 1, 1; 0, 1; 2, 14; 2, 1;)$
146. $x^3+5x-9=0$	$(61,12,9)$	$\epsilon'=\epsilon^2$	$(1, 6; 2, 3; \mid 0, 3; 0, 10; 0, 3; 0, 3; 3, 8; 3, 3;)$
147. $x^2-5x-9=0$	$(22,20,7)$		$(2, 2; 1, 1; \mid 0, 2; 1, 2; 0, 1; 3, 13; 3, 1;)$
148. $x^3+6x-9=0$	$(865,140,116)$	$\epsilon'=\epsilon^2$	$(1, 7; 2, 3; \mid 8, 4; 0, 1; 0, 1; 0, 4; (12), (5); (12), \cdots)$
149. $x^3-6x-9=0$	reducible.		
150. $x^3+7x-9=0$	$(8,1,1)$		$(1, 8; 2, 10; \mid 3, 10;)$
151. $x^3-7x-9=0$	$(20,22,7)$		$(3, 3; -1, 7; \mid 2, 22; 2, 7;)$
152. $x^3+8x-9=0$	reducible.		
153. $x^3-8x-9=0$	$(2,1,0)$	$\epsilon'=\epsilon^5$	$(3, 2; 2, 2; \mid 2, 1; 1, 2; 2, 2; 1, 1; 2, 2; 5, 21; 5, 2;)$
154. $x^3+9x-9=0$	$(10,1,1)$		$(0, 9; 0, 1; 0, 1;)$
155. $x^3-9x-9=0$	$\epsilon_1(13,17,5)$	$\epsilon'_1=\epsilon_1^2$	$(3, 2; 1, 2; \mid 0, 1; 1, 1; (6), (20); (6), \cdots)$
		or $\epsilon'_1=\epsilon_1$	$(3, 3; -2, 2; \mid 0, 1; 0, 2; 1, 24; 1, 2;)$
	$\epsilon_2(-8,-1,1)$		$(-1, -8; -2, -6; \mid -3, -6;)$
$\epsilon_1\epsilon_2^2\epsilon_3=1$	$\epsilon_3(16,9,-4)$		$(-2, -4; 0, -4; \mid -2, 5; -2, -4;)$
156. $x^3+9x-1=0$	$(9,0,1)$		$(0, 9;)$
157. $x^3-9x-1=0$	$\epsilon_1(0,3,1)$		$(3, 0; 6, 18; \mid 9, 18;)$
	$\epsilon_2(-9,0,1)$		$(0, -9;)$
	$\epsilon_3(0,-3,1)$		$(-3, 0; -6, 18; \mid -9, 18;)$
158. $x^3+9x-2=0$	$(1475017, 36028, 163006)$		$(0, 9; 0, 4; 2, 4; 1, 2; 1, 2; 0, 1; 0, 1; 0, 1; (0), (1); (0), \cdots)$
159. $x^3-9x-2=0$	$\epsilon_1(12479573, 60177508, 19377828)$		
	$\epsilon_2(161,4,-18)$	$\epsilon'_2=\epsilon_2^4$	$(0, -8; 4, -4; \mid -2, 4; 0, 8; -4, -4; 2, -4; (0), (-9); (0), \cdots)$
	$\epsilon_3(-59,-245,85)$		$(-3, -1; 2, 6; \mid 4, 1; 4, 6; -1, 16; -1, 6;)$

160. $x^3+9x-3=0$ (82, 3, 9) (0, 9; 0, 3; 0, 3;)

161. $x^3-9x-3=0$ ϵ_1(37, 123, 39) (3, 0; 6, 6; | 3, 6; 9, 18; 9, 6;)

ϵ_2(—80, —3, 9) (0, —9; 0, —3; 0, —3)

$\epsilon_1\epsilon_2^2\epsilon_3^3=1.$ ϵ_3(1, 3, —1) (—2, 1; 0, —1; | —2, 11; —2, —1;)

162. $x^3+9x-4=0$ (57387437, 2718291, 6244913)

(0, 8; | 2, 2; 0, 1; 0, 2; 0, 1; 0, 1; 4, 4; 1, 2; 1, 1; 0, 2; 3, (3); (0), (3); · · ·)

163. $x^3-9x-4=0$ ϵ_1(25, 64, 20) (3, 1; 1, 4; | 4, 4; 4, 20; 4, 4;)

ϵ_2(—4291, —222, 488) (—1, —9; 0, 1; | 2, 2; 3, 4; 5, 4; 3, 2; 2, 1; —1, 8; —1, 1;)

ϵ_3(115, 217, —79) *NE*(—3, —1; —2, —4; | 0, —5; 0, —4; —5, 16; —5, —4;)

164. $x^3+9x-5=0$ (6317, 366, 680) (0, 9; 0, 1; 1, 1; 1, 1; 1, 2; 0, 1; (1), (1); (1), · · ·)

165. $x^3-9x-5=0$ ϵ_1(31, 65, 20) (3, 1; 2, 5; | —5, 5; 5, 20; 5, 5;)

ϵ_2(—43, —3, 5) (—1, —8; —2, 2; | 1, 2; —3, —6; —3, 2;)

$\epsilon_1\epsilon_2\epsilon_3=1.$ ϵ_3(—17, —24, 9) $\epsilon'_3=\epsilon_3^2$

(—3, —2; 0, 3; | 0, 3; —6, 16; —6, 3; 0, 3; —3, 14; —3, 3;)

166. $x^3+9x-6=0$ (3489013, 236574, 370902)

(0, 9; 0, 1; 0, 1; 0, 1; 0, 1; 2, 4; 1, 1; 0, 3; 3, 2; 2, 1; 0, (1); (0), (1); · · ·)

167. $x^3-9x-6=0$ ϵ_1(1, 3, 1)

ϵ_2(289, 24, —34) $\epsilon'_2=\epsilon_2^2$

(—1, —8; —1, 3; | —1, —2; 1, 2; 1, 3; 0, —8; 0, 3; 1, 2; 1, —2; 1, 3; —2, —6; —2, 3;)

$\epsilon_1^2\epsilon_2\epsilon_3=1.$ ϵ_3(469, 522, —202) $\epsilon'_3=\epsilon_3^2$

NE(—2, —2; 1, 2; | 1, 2; 0, 2; —1, 2; 1, 3; 1, —2; 1, —2; (1), (9); (1), · · ·)

168. $x^3+9x-7=0$ (143, 11, 15) (0, 8; | 2, 1; 3, 8; 3, 1; 2, 7;)

169. $x^3-9x-7=0$ ϵ_1(19, 30, 9) $\epsilon'=\epsilon^2$ (3, 2; 0, 3; | 0, 3; 6, 20; 6, 3; 0, 3; 3, 22; 3, 3;)

ϵ_2(—8, —1, 1) *NE*(—1, —8; —2, —6; | —3, —6;)

$\epsilon_1\epsilon_2^3\epsilon_3=1$ ϵ_3(—17, —15, 6) (—3, —3; 0, 3; | —3, 3; —3, 12; —3, 3;)

170. $x^3+9x-8=0$ (308121, 26292, 31822)

(0, 9; 0, 1; 0, 1; 0, 3; 1, 1; 1, 8; 0, 1; (2), (5); (2), · · ·)

171. $x^3-9x-8=0$ reducible.

172. $x^3-10=0$ (181, 84, 39) (2, 4; 4, 6; | 3, 6; 6, 12; 6, 6;)

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Triads of Plane Curves.*

By A. R. Jerbert.

1. *Introduction.* Papers have recently appeared by Professor E. P. Lane and A. F. Carpenter on ruled surfaces with generators in correspondence.† The following paper considers a somewhat similar problem, namely that of three plane curves with points in correspondence.

If the correspondence is such that the corresponding points are not collinear, we find that a projective theory of the configuration can be based upon a system of three ordinary linear first order differential equations.

In investigating the correspondence between the given curves it proves advantageous to introduce two associated triads of curves. They are the curves enveloped by the sides of the triangle formed by joining corresponding points of the given curves and the curves generated by the points in which the tangents to the given curves meet the opposite sides of this same triangle. We have called these curves the contact curves and tangential curves respectively.

A special case consists in showing that if the coefficients in the defining system of differential equations are constants the curves defined by the system, together with the associated contact and tangential curves, constitute a set of nine equiangular, logarithmic spirals.

2. *The Differential Equations.* Let the three homogeneous coördinates of an arbitrary point P_x on a curve C_x be given as analytic functions of an independent variable t. And in like manner let the three coördinates of each of two other points P_y and P_z be given as functions of the same variable t. Then, as t varies, the points P_x, P_y, P_z, describe three plane curves whose points are in correspondence. The object of this paper is to discover some of the implications of this correspondence.

We shall suppose that the points P_x, P_y, P_z, are not collinear. Then the determinant

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* Presented to the Society, San Francisco Section, June 18, 1927.

† E. P. Lane, "Ruled Surfaces with Generators in One to One Correspondence," *Transactions of the American Mathematical Society*, Vol. 25 (1923); A. F. Carpenter, "Triads of Ruled Surfaces," *Transactions of the American Mathematical Society*, Vol. 29 (April 1927).

$$\text{(D)} \qquad \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

is different from zero, and it is possible to determine the coefficients in the system of differential equations

$$\text{(1)} \qquad \begin{aligned} x' &= a_1x + b_1y + c_1z, \\ y' &= a_2x + b_2y + c_2z, \\ z' &= a_3x + b_3y + c_3z, \end{aligned}$$

so that (x_i, y_i, z_i) $(i = 1, 2, 3)$ will be three sets of solutions. For example, we may substitute each of these three sets in turn in the first of system (1) and then solve the resulting three equations for the coefficients of the first of equations (1). Similarly we may determine the coefficients of each of the other three equations.

3. *A Canonical System.* If we have constructed a defining system (1) based on three points $(x)=(x_1, x_2, x_3)$, $(y)=(y_1, y_2, y_3)$, $(z)=(z_1, z_2, z_3)$, the question arises—what will be the effect on the coefficients in (1) if we multiply the x's by an arbitrary function $\lambda(t)$ the y's by $\mu(t)$, and the z's by $\nu(t)$? The factors λ, μ, ν, will not affect the points (x), (y), (z), since we are dealing with homogeneous coördinates, but they will affect the coefficients in the resulting system (1).

Accordingly, in (1) we shall put

$$\text{(2)} \qquad x = \lambda\bar{x}, \quad y = \mu\bar{y}, \quad z = \nu\bar{z}.$$

System (1) is transformed by (2) into

$$\text{(3)} \qquad \begin{aligned} \bar{x}' &= (a_1 - \lambda'/\lambda)\bar{x} + (b_1\mu/\lambda)\bar{y} + (c_1\nu/\lambda)\bar{z}, \\ \bar{y}' &= (a_2\lambda/\mu)\bar{x} + (b_2 - \mu'/\mu)\bar{y} + (c_2\nu/\mu)\bar{z}, \\ \bar{z}' &= (a_3\lambda/\nu)\bar{x} + (b_2\mu/\nu)\bar{y} + (c_3 - \nu'/\nu)\bar{z}. \end{aligned}$$

If we place

$$\lambda = k_1 \exp \int a_1dt, \quad \mu = k_2 \exp \int b_2dt, \quad \nu = k_3 \exp \int c_3dt,$$

then system (3) takes the canonical form

$$\text{(4)} \qquad \bar{x}' = \bar{b}_1\bar{y} + \bar{c}_1\bar{z}, \quad \bar{y}' = \bar{a}_2\bar{x} + \bar{c}_2\bar{z}, \quad \bar{z}' = \bar{a}_3\bar{x} + \bar{b}_3\bar{y}$$

where

$$\text{(5)} \qquad \begin{aligned} \bar{b}_1 &= (k_2/k_1)\, b_1 \exp \int (b_2 - a_1)\,dt, & \bar{c}_1 &= (k_3/k_1)c_1 \exp \int (c_3 - a_1)\,dt, \\ \bar{a}_2 &= (k_1/k_2)\, a_2 \exp \int (a_1 - b_2)\,dt, & \bar{c}_2 &= (k_3/k_2)c_2 \exp \int (c_3 - b_2)\,dt, \\ \bar{a}_3 &= (k_1/k_3)a_3 \exp \int (a_1 - c_3)\,dt, & \bar{b}_3 &= (k_2/k_1)b_3 \exp \int (b_2 - c_3)\,dt, \end{aligned}$$

It can be verified that the canonical form (4) is unique except for the arbitrary constants k_1, k_2, k_3. It is also evident that the canonical form is preserved under a λ-transformation in which λ, μ, ν, are any three constants. All of this is geometrically intuitive if we consider that a λ-transformation shifts the points (x'), (y'), (z'), since with $x = \lambda\bar{x}$ we have $x' = \lambda'\bar{x} + \lambda\bar{x}'$. This shift reduces to zero if $\lambda = \text{const.}$

The meaning of the canonical form (4), expressed geometrically, is that the derivative points corresponding to the vertices of the triangle (x), (y), (z) lie respectively in the opposite sides (yz), (zx), (xy).

It is to be noted that in the uncanonical form (1), the derivative points (x'), (y'), (z') were devoid of geometrical significance aside from the fact that they, together with the points (x), (y), and (z), respectively determined the tangent lines to the curves C_x, C_y, and C_z. In the canonical system (4) the derivative points are unique points on the tangents, namely, they are the points where the tangents to C_x, C_y, and C_z meet the opposite sides of the triangle (x), (y), (z).

In the uncanonical system (1) these points are given by the expressions

$$b_1y + c_1z,\quad a_2x + c_2z,\quad a_3x + b_3y. \tag{6}$$

The covariant character of the points given by (6) may readily be verified by means of equations (2) and (3). We shall call them the tangential points, and the curves generated by them, the tangential curves.

4. *Complete System of Invariants.* The complete system of invariants for the defining system (1) is readily computed by the well known Lie method of continuous transformations. The transformations, in this case, are the λ, μ, ν, transformations already mentioned and utilized in obtaining the canonical form (4).

The transformation on the parameter t is of no interest in this connection because under it the coefficients in the defining system (1) are merely multiplied by a factor. As a result, the functions of the coefficients which are invariant under the λ, μ, ν, transformation are relative invariants under the transformation of parameter.

If we consider invariant functions of the coefficients which are free from derivatives it develops that there are four which are independent. They may be taken to be any four of the following five,

$$A = b_3c_2,\quad B = c_1a_3,\quad C = b_1a_2,\quad H = a_2b_3c_1,\quad G = a_3b_1c_2. \tag{7}$$

The difference and the sum of the last two will prove to be of some importance. They are

$$(8)\qquad \begin{aligned}\Delta_1 &= H - G = a_2 b_3 c_1 - a_3 b_1 c_2,\\ \Delta_2 &= H + G = a_2 b_3 c_1 + a_3 b_1 c_2.\end{aligned}$$

If we consider invariants involving derivatives we find that there are 10 which are independent. Eight of these are accounted for by the original four together with the derivatives of these four. There remain two invariants to be found. For the latter we can take any two of the following three,

$$(9)\qquad \begin{aligned}D_1 &= a_1 - b_2 + (c_2/c_1)'(c_1/c_2),\\ D_2 &= b_2 - c_3 + (a_3/a_2)'(a_2/a_3),\\ D_3 &= c_3 - a_1 + (b_1/b_3)'(b_3/b_1).\end{aligned}$$

Between the invariants (9) and those given in (7), there exists the following relation,

$$G/H = \exp \int (D_1 + D_2 + D_3)\,dt.$$

At this stage our system of invariants is complete in the sense that all other invariants of whatever order in the derivatives, may be obtained from the ones already gotten, by algebraic processes and by differentiation.

With respect to the covariants the discussion is simplified by the fact that there is no occasion to search for covariants involving derivatives of the variables, the latter being expressible in terms of the variables themselves by virtue of the defining system of differential equations.

A complete system of covariants is obtained by adjoining to the invariants already noted, any two of the following six,

$$a_2 x/y,\ b_1 y/x,\ a_3 x/z,\ c_1 z/x,\ b_3 y/z,\ c_2 z/y.$$

It is to be noted that the coefficients in the defining system (1) are relative invariants whose vanishing, in each case, has an obvious geometrical significance.

5. *Dual Considerations.* In proceeding with the Geometry of the triple of curves, we shall first find the adjoint system corresponding to the system (4). To this end, denote the line y, z as (u). The line coördinates of (u) i. e. (u_1, u_2, u_3), can be taken as any three numbers proportional to the cofactors of the x's in the determinant Δ (equation D). We shall take the cofactors themselves, thus making a choice of proportionality factor. In similar fashion the lines z, x and x, y are denoted by (v) and (w) respectively with the (v_1, v_2, v_3) and the (w_1, w_2, w_3) given by the cofactors of the y's and z's respectively, in Δ.

We find by direct computation that the adjoint system is

$$
\begin{array}{lrrr}
 & u' = & & -a_2 v \; -a_3 w, \\
(10) & v' = & -b_1 u & -a_3 w, \\
 & w' = & -c_1 u - c_2 v & .
\end{array}
$$

The adjoint system corresponding to the uncanonical defining system (1) is the same as (10) with the diagonal terms involving a_1, b_2, c_3, restored.

In interpreting the adjoint system (10) it is important to keep in mind the fact that just as the join of x and x' is the tangent at (x), to the curve C_x traced by the point (x), so the intersection of (u) and (u') is the contact point on (u) i. e. the point where (u) touches its envelope curve. In the same manner $(v') = (v'_1, v'_2, v'_3)$, is a line through the contact point on (v) while (w') is a line through the contact point on (w). From the form of system (27) we see that the lines (u'), (v'), (w') pass respectively through the opposite vertices (x), (y), and (z). We shall refer to (u'), (v'), (w'), as the "contact lines."

Incidentally, by finding the intersections of (u), (u'); (v), (v'); (w), (w'); we learn that the contact points are given by the expressions

$$
\begin{array}{lrrr}
 & \tau_x = & & a_3 y - a_2 z, \\
(11) & \tau_y = & -b_3 x & + b_1 z, \\
 & \tau_z = & c_2 x - c_1 y & .
\end{array}
$$

6. *Theorems involving Tangentials and Contact Points.* We shall denote by ${}_h\tau_x$ the harmonic conjugate of the contact point τ_x with respect to the points (y) and (z). In the same manner ${}_h\tau_y$ and ${}_h\tau_z$ denote the harmonic conjugates of the other two contact points with respect to z, x and x, y respectively.

We obtain the expression for ${}_h\tau_y$ from the expression for τ_x in (11) by changing the sign of the coefficient of z. A similar change of sign converts τ_y and τ_z into ${}_h\tau_y$ and ${}_h\tau_z$ respectively. We find accordingly,

$$
\begin{array}{lrrr}
 & {}_h\tau_x = & & a_3 y + a_2 z, \\
(12) & {}_h\tau_y = & b_3 x & + b_1 z, \\
 & {}_h\tau_z = & c_2 x + c_1 y & .
\end{array}
$$

In a similar manner we indicate the harmonic conjugates of the tangentials x', y', and z' with respect to y, z; z, x; and x, y respectively, by ${}_hx'$, ${}_hy'$ and ${}_hz'$ and find

$$
\begin{array}{lrrr}
 & {}_hx' = & & b_1 y - c_1 z, \\
(13) & {}_hy' = & -a_2 x & + c_2 z, \\
 & {}_hz' = & a_3 x - b_3 y & .
\end{array}
$$

If we designate by $\Delta_{x'}$ the determinant of the tangentials x', y', z', and by Δ_τ the determinant of the harmonic conjugates of the contact points, we find from (6) and (12)

$$\begin{aligned} \Delta_{x'} &= \Delta\ \Delta_2, \\ \Delta_\tau &= \Delta\ \Delta_2, \end{aligned} \tag{14}$$

where Δ is the determinant of the points x, y, z (equation D), and Δ_2 is the determinant defined by (8).

Since the vanishing of Δ is precluded, we have from (14)

THEOREM 1. *If the tangentials are collinear so also are the harmonic conjugates of the contact points and conversely. The necessary and sufficient condition for this is expressed by*

$$\Delta_2 = 0.^*$$

Similar statements can be made in connection with the vanishing of the invariant Δ_1.

It will prove of interest to determine under what conditions the tangentials given by (16) are harmonic respectively to the contact points (11) and pairs of vertices of the triangle (x), (y), (z). The conditions are

$$\begin{aligned} C &= B, \\ A &= C, \\ B &= A, \end{aligned} \tag{15}$$

where A, B, and C are defined in (7).

Equation $(15)_1$ is the condition that the tangential $b_1 y + c_1 z$ shall be harmonic to the contact point $\tau_x = a_3 y - a_2 z$. Similar statements hold for $(15)_2$ and $(15)_3$. Since any two of the equalities (15) imply the third we have

THEOREM 2. *If any two of the tangentials are harmonic with respect to the corresponding contact points, then the third tangential is also harmonic with respect to its corresponding contact point.*

Theorems 1 and 2 have supplied geometric interpretations for the relative invariants A, B, C, Δ_1, and Δ_2. Interpretations for D_1, D_2, D_3, appear in the following situation. Denote by X, Y, and Z the contact points on the

* From the known properties of triangles we can infer in theorem 1 that with the vanishing of Δ_2 not only will the tangentials be collinear but the harmonic conjugates of the tangent lines will be concurrent. In the same manner the collinearity of the contact point conjugates carries with it the concurrency of the contact lines.

lines joining τ_x, τ_y, τ_z, to the opposite vertices (x), (y) and (z). The following statements hold true.

$$
(16) \qquad \begin{array}{ll}
\Delta_2 = D_1(A),\ yZ \to \tau_y, & \Delta_2 = -D_2(B),\ yX \to \tau_y. \\
\Delta_2 = D_2(B),\ zX \to \tau_z, & \Delta_2 = -D_3(C),\ zY \to \tau_z. \\
\Delta_2 = D_3(C),\ xY \to \tau_x, & \Delta_2 = -D_1(A),\ xZ \to \tau_x.
\end{array}
$$

The notation is most readily understood by expanding the first statement of (16).

If and only if, $\Delta_2 = D_1(A)$, *will the line joining* (y) *to* (Z) *meet the opposite side of the triangle* (x), (y), (z) *in the point* τ_y.

7. *Defining Systems for the Tangential and Contact Curves.* By differentiating the expressions for the contact points given in (11) and eliminating x', y', z', by (4) and x, y, z by (11), we obtain expressions for τ_x', τ_y', τ_z', in terms of τ_x, τ_y, and τ_z. That is, we obtain for the contact points a defining system similar to (1).

By means of (7), (8) and (9) we can express the coefficients of τ_x, τ_y, τ_z, in terms of the fundamental invariants. The defining system is accordingly,

$$
(17) \qquad \begin{aligned}
\tau_x' &= A_1\tau_x + B_1\tau_y + C_1\tau_z, \\
\tau_y' &= A_2\tau_x + B_2\tau_y + C_2\tau_z, \\
\tau_z' &= A_3\tau_x + B_3\tau_y + C_3\tau_z.
\end{aligned}
$$

where

$$
(18) \qquad \begin{array}{ll}
-b_1\Delta_1B_1 = a_2G(D_2 + G/C - H/B), & -c_1\Delta_1C_1 = a_3H(D_2 + G/C - H/B), \\
-a_2\Delta_1A_2 = b_1H(D_3 + G/A - H/C), & -c_2\Delta_1C_2 = b_3G(D_3 + G/A - H/C), \\
-a_3\Delta_1A_3 = c_1G(D_1 + G/B - H/A), & -b_3\Delta_1B_3 = c_2H(D_1 + G/B - H/A).
\end{array}
$$

From (35), we see that the following statements are true.

$$
(19) \qquad \begin{array}{lll}
D_2 + G/C - H/B = 0, & B_1 = C_1 = 0, & \tau_x = k_1, \\
D_3 + G/A - H/C = 0, & A_2 = C_2 = 0, & \tau_y = k_2. \\
D_1 + G/B - H/A = 0, & A_3 = B_3 = 0, & \tau_z = k_3.
\end{array}
$$

The complete statement corresponding to $(19)_1$, is—*the vanishing of the invariant* $D_2 + G/C - H/B$ *is the necessary and sufficient condition that the contact curve* τ_x *should degenerate into a point.*

For the sake of completeness the coefficients in the defining systems for the curves traced by the tangentials, tangential harmonics, and contact harmonics have been computed. They are as follows:

Tangentials.

$$
\begin{aligned}
c_2\Delta_2 B_1 &= c_1 G(E_2 + G/B - H/C + AC/G + AB/G),\\
b_3\Delta_2 C_1 &= -\, b_1 H(E_2 + G/B - H/C - AB/H - AC/H),\\
c_1\Delta_2 A_2 &= -\, c_2 H(E_3 + G/C - H/A - AB/H - BC/H),\\
a_3\Delta_2 C_2 &= a_2 G(E_3 + G/C - H/A + AB/G + BC/G),\\
b_1\Delta_2 A_3 &= b_3 G(E_1 + G/A - H/B + AC/C + BC/C),\\
a_2\Delta_2 B_3 &= -\, a_3 H(E_1 + G/A - H/B - AC/H - BC/H).
\end{aligned}
$$

Tangential Harmonics.

$$
\begin{aligned}
c_2\Delta_1 B_1 &= c_1 G(E_2 + H/C - G/B - AC/G + AB/G),\\
b_3\Delta_1 C_1 &= b_1 H(E_2 + H/C - G/B - AC/H + AB/H),\\
c_1\Delta_1 A_2 &= c_2 H(E_3 + H/A - G/C - AB/H + BC/H),\\
a_3\Delta_1 C_2 &= a_2 G(E_3 + H/A - G/C - AB/G + BC/G),\\
b_1\Delta_1 A_3 &= b_3 G(E_1 + H/B - G/A - BC/G + AC/G),\\
a_2\Delta_1 B_3 &= a_3 H(E_1 + H/B - G/C - BC/H + AC/H).
\end{aligned}
$$

Contact Harmonics.

$$
\begin{aligned}
b_1\Delta_1 B_1 &= -\, a_2 G(D_2 + H/B - G/C - 2H/A),\\
c_1\Delta_1 C_1 &= a_3 H(D_2 + H/B - G/C + 2G/A),\\
a_2\Delta_2 A_2 &= b_1 H(D_3 + H/C - G/A + 2G/B),\\
c_2\Delta_1 C_2 &= -\, b_3 G(D_3 + H/C - G/A - 2H/B),\\
a_3\Delta_1 A_3 &= -\, c_1 G(D_1 + H/A - G/B - 2H/C),\\
b_3\Delta_2 B_3 &= c_2 H(D_1 + H/A - G/B + 2G/C).
\end{aligned}
$$

8. *System of Linear Differential Equations with Constant Coefficients.* If the coefficients in system (1) are constants we can assume the solutions—

$$
(20)\qquad
\begin{aligned}
x &= a e^{rt},\\
y &= \beta e^{rt},\\
z &= \gamma e^{rt}.
\end{aligned}
$$

where a, β, γ are to be determined. On substituting these values in (1) and suppressing the factor e^{rt} we find

$$
(21)\qquad
\begin{aligned}
(a_1 - r)\, a + b_1\beta + c_1\gamma &= 0,\\
a_2 a + (b_2 - r)\,\beta + c_2\gamma &= 0,\\
a_3 a + b_3\beta + (c_3 - r)\gamma &= 0.
\end{aligned}
$$

For these equations to be consistent the determinant of the coefficients must vanish. The resulting cubic in r is satisfied, in general, by three distinct values $r = r_i$ $(i = 1, 2, 3)$. Substituting in turn each value of r in

(39) we obtain the corresponding values of α, β, γ. In this way we find the three sets of solutions

$$
(22) \qquad
\begin{aligned}
(x) &\equiv x_1 = \alpha_1 e^{r_1 t}, & x_2 &= \alpha_2 e^{r_2 t}, & x_3 &= \alpha_3 e^{r_3 t}, \\
(y) &\equiv y_1 = \beta_1 e^{r_1 t}, & y_2 &= \beta_2 e^{r_2 t}, & y_3 &= \beta_3 e^{r_3 t}, \\
(z) &\equiv z_1 = \gamma_1 e^{r_1 t}, & z_2 &= \gamma_2 e^{r_2 t}, & z_3 &= \gamma_3 e^{r_3 t}.
\end{aligned}
$$

Curves (x), (y), and (z) are equiangular logarithmic spirals. They possess the further property noted by Klein * of being transformable into themselves by the transformation

$$
(\mathrm{T}) \qquad
\begin{aligned}
\bar{x}_1 &= e^{r_1\lambda} x_1, \\
\bar{x}_2 &= \qquad e^{r_2\lambda} x_2, \\
\bar{x}_3 &= \qquad\qquad e^{r_3\lambda} x_3.
\end{aligned}
$$

If λ is regarded as a continuous variable then T constitutes a closed system of commutative linear transformations which carry a point of either (x), (y) or (z) with parameter t into a point of the same curve with parameter $t+\lambda$.

A set of curves transformable into themselves by a system T, has been named by Klein W-curves of a system.† The curves (x), (y), and (z) possess the further well known property of being projectively transformable into each other.

If we recall the expression (6) and (11) for the tangentials and the contact points we see that they are in the present discussion, constant linear combinations of the points (x), (y), and (z). From (22) it follows that the tangentials and the contact points describe logarithmic equiangular spirals belonging to the same system as the curves (x), (y) and (z), i. e., they are equiangular with the latter curves.

We can conclude the foregoing results in the

THEOREM 3. *A system of three first order linear homogeneous differential equations with constant coefficients defines, in general, three equiangular, logarithmic spirals under such a correlation, by virtue of the common parameter, that the associated tangential and contact curves are also logarithmic spirals equiangular with the given curves. The curves themselves and the associated tangential and contact curves are all W-curves of a system in Klein's sense.*

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* *Mathematische Annalen*, Vol. 4 (1871), p. 51: "Über diejenigen ebenen Curven, welche durch ein geschlossenes system von einfach unendlich vielen vertauschbaren, linearen, Transformationen in sich übergehen."

† See reference following equation (22).

We can summarize the present paper by stating that a system of three linear, homogeneous, first order, differential equations in one independent and three dependent variables defines three plane curves to within a projectivity. Furthermore, there is established between the curves by virtue of their expression in terms of a common parameter a point correspondence.

The properties of this triad of curves are a function, not alone of the curves themselves, but of the curves and the correlation established between them.

In conclusion we shall note a few of the problems which suggest themselves in connection with the present paper.

1. We have found the condition that the contact points be collinear, namely $\Delta_2 = 0$. We might ask what is the line locus described by the line of contact points when the condition $\Delta_2 = 0$, is fulfilled?

2. There is the obvious possibility of extending all of these results first to four equations in four dependent variables interpretable as defining four curves in three dimensions, and finally to n-equations which will define n curves in $(n - 1)$ space.

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The Symbolic Development of the Disturbing Function.

By K. P. Williams.

1. In the development of the disturbing function, as carried out by LeVerrier, one becomes bewildered by the number of series with which he must deal. The method is elementary in principle, but extremely laborious and confusing in practice. The problem is preëminently one for which some symbolic method is desirable. This seems to have been first realized by Newcomb, who gave in the *American Journal of Mathematics* the first description of his method.* Later he published in detail, with the necessary numerical tables, a symbolic development in terms of the eccentric anomaly.† In practice the development in terms of the eccentric anomaly did not prove to possess the advantages that the author had anticipated, and he subsequently gave a briefer symbolic development in terms of the mean anomaly, which he used in an investigation of the secular variations of the four inner planets.‡

2. In the present paper a modification of Newcomb's method will be developed, which appears to the writer to possess certain distinct advantages. A group of fundamental operators are clearly isolated, and the development is effected by means of them. Incidentally certain symbols are introduced which possess a simple algebra, and the employment of them does away with the use of trigonometric functions, and their continual combination. The latter are reintroduced only at the end. The development proceeds according to the mean anomaly.

3. Let γ be the angle between the planes of the orbits of the two planets, and λ and λ' be the angular distances of the planets from their common node. Put $\sigma = \sin \frac{1}{2}\gamma$, $\alpha = a/a'$, where a and a' are the semi-major axes of the two orbits. Let $c_n^{(i)}$, $(n = 1, 3, 5, 7;\ i = 0, 1, 2, \cdots)$ be the constants called by Newcomb the *Gaussian coefficients.*§ They are related to the familiar coefficients $b_n^{(i)}$ used by LeVerrier, by the equation

* Vol. 3, pp. 193-209.

† *Astronomical Papers of the American Ephemeris,* Vol. 3, pp. 1-200.

‡ *Astronomical Papers,* Vol. 5, pp. 1-48, pp. 301-378.

§ *Loc. cit.,* Vol. 5, p. 339 ff.

$$c_n^{(i)} = \alpha^{(n-1)/2}\, b_n^{(i)}.$$

Let

$$A_i = c_1^{(i)} - (1/2)\sigma^2(c_3^{(i-1)} + c_3^{(i+1)}) + (3/8)\,\sigma^4(c_5^{(i+2)} + 4c_5^{(i)} + c_5^{(i-2)})$$
$$-(5/16)\sigma^6(c_7^{(i+3)} + 9c_7^{(i+1)} + 9c_7^{(i-1)} + c_7^{(i-3)}),$$
$$B_i = (1/2)c_3^{(i)} - (3/4)\sigma^2(c_5^{(i+1)} + c_5^{(i-1)})$$
$$+ (15/16)\sigma^4(c_7^{(i+2)} + 3c_7^{(i)} + c_7^{(i-2)}),$$
$$C_i = (3/8)c_5^{(i)} - (15/16)\sigma^2(c_7^{(i+1)} + c_7^{(i-1)}),$$
$$D_i = (5/16)c_7^{(i)}.$$

Following Newcomb, put

$$V_i = i\lambda' - i\lambda,$$
$$V_i^{(1)} = (i+1)\lambda' - (i-1)\lambda,$$
$$V_i^{(2)} = (i+2)\lambda' - (i-2)\lambda,$$
$$\cdots\cdots\cdots\cdots$$

Then, if Δ_0 denotes the distance between the two planets, with the eccentricities neglected, we have *

$$a'\Delta_0^{-1} = \sum_{i=-\infty}^{\infty} (1/2)A_i \cos V_i \quad \text{(called terms of class 0)}$$
$$+ \sigma^2 B_i \cos V_i^{(1)} \quad \text{(called terms of class 1)}$$
$$+ \sigma^4 C_i \cos V_i^{(2)} \quad \text{(called terms of class 2)}$$
$$\cdots\cdots$$

It is seen that we can write

(1) $$\Delta_0^{-1} = \sum N \cos(\nu\lambda' + \mu\lambda),$$

with the following identifications:

Class	N	μ	ν
0	$(1/2)A_i/a'$	$-i$	i,
1	$\sigma^2 B_i/a'$	$-i+1$	$i+1$,
2	$\sigma^4 C_i/a'$	$-i+2$	$i+2$,
3	$\sigma^6 D_i/a'$	$-i+3$	$i+3$,
. . .	. . .	. . .	. . .

Next let

g and g' be the mean anomalies,
f and f' be the true anomalies,
y and y' be the equations of the centers,
ω and ω' be the distances from the common node to the perihelia.

* Newcomb, *loc. cit.*, Vol. 3, pp. 14, 15.

Then

$$\lambda = \omega + f = \omega + g + (f - g) = \omega + g + y,$$
$$\lambda' = \omega' + g' + y'.$$

The quantity $\cos(\nu\lambda' + \lambda)$ in (1) then becomes

$$\cos(\nu\omega' + \mu\omega + \nu(g' + y') + \mu(g + y)),$$

which may be obtained by replacing g and g' by $g + y$ and $g' + y'$, respectively, in

$$\cos(\nu\omega' + \mu\omega + \nu g' + \mu g).$$

It will be seen that when the expansions are being made we can omit ω and ω', since it is possible to restore them in the final results. We thus work with

$$N \cos(\nu g' + \mu g).$$

When the eccentricities are to be taken into account we have merely to replace a, a', g, and g' by r, r', $g + y$, and $g' + y'$ respectively. Since r and r', as well as y and y' can be expressed in terms of the mean anomalies and the eccentricities, it is clear that an expansion in terms of the eccentricities and the mean anomalies can be found for Δ^{-1} where Δ is the distance between the two planets. The problem is clearly so complex that a symbolic treatment is desirable.

We first suppose that the eccentricity of the outer orbit is zero. Nothing is therefore to be done to a' and g', and accordingly they need not be exhibited. We therefore finally reduce the problem to the consideration of the change produced in

$$N \cos \mu g,$$

when the eccentricity of the inner orbit is introduced.

4. It is characteristic of Newcomb's method to regard N as a function of $\log a$ and $\log a'$. Remembering that the $c_n^{(i)}$ are all functions of $\alpha = a/a'$, and omitting the quantity σ, we have

$$N = (1/a')\, \phi\, (a/a').$$

It is easy to show that

$$\partial N/\partial \log a + \partial N/\partial \log a' = -N.$$

Therefore if we introduce the operators $\mathfrak{D}$ and $\mathfrak{D}'$ defined by

$$\mathfrak{D} = \partial/\partial \log a, \mathfrak{D}' = \partial/\partial \log a',$$

we have the simple operative relation

(2) $$\mathfrak{D} + \mathfrak{D}' = -1, \quad \text{or} \quad \mathfrak{D}' = -1 - \mathfrak{D}.$$

If further we let

$$D = \partial/\partial \log \alpha,$$

we can show that

(3) $$\mathfrak{D}\, N = DN,$$

and therefore by (2), it follows that

(4) $$\mathfrak{D}' N = -(1 + D)N.$$

It may be remarked that Newcomb's development of the Gaussian constants gave methods of obtaining

$$D^m A_i, \quad D^m B_i, \quad D^m C_i, \quad D^m D_i,$$

so that the results of operating with $\mathfrak{D}$ and $\mathfrak{D}'$ upon N can be regarded as known.*

5. We turn now to a consideration of the analytic problem involved. We first develop a form for the nth derivative of a function of a function which is specially adapted to the present problem.

Suppose we define $F(t)$ from functions $G(x, t)$, $x = \phi(t)$, by putting

$$F(t) = \partial G/\partial x,$$

having placed $x = \phi(t)$ everywhere in the derivative $\partial G/\partial x$. It is seen that we can write

$$F'(t) = \frac{\partial}{\partial x} \frac{dG}{dt}, \quad F''(t) = \frac{\partial}{\partial x} \frac{d^2 G}{dt^2}, \quad F'''(t) = \frac{\partial}{\partial x} \frac{d^3 G}{dt^3}, \cdots$$

provided that dG/dt, d^2G/dt^2, $\cdots$, are all left properly expressed in terms of x and t.

Next consider

$$z = f(x), \quad x = \phi(t).$$

We have

$$\frac{dz}{dt} = \frac{\partial}{\partial x}\left(z \frac{dx}{dt}\right).$$

If we let $z \dfrac{dx}{dt}$ be the function $G(x, t)$ above, we have

$$\frac{d^{n+1} z}{dt^{n+1}} = \frac{\partial}{\partial x}\left[\frac{d^n}{dt^n}\left(z \frac{dx}{dt}\right)\right].$$

---

* *Loc. cit.*, Vol. 3, pp. 42-74.

Applying the formula of Leibnitz we obtain

$$(5)\qquad \frac{d^{n+1}z}{dt^{n+1}} = \frac{\partial}{\partial x}\left[\frac{d^n z}{dt^n}\,\frac{dx}{dt} + n\,\frac{d^{n-1}z}{dt^{n-1}}\,\frac{d^2x}{dt^2} + \cdots + z\,\frac{d^{n-1}x}{dt^{n+1}}\right],$$

the coefficients being the binomial coefficients.

In using (5) it is necessary to leave $dz/dt, \cdots, d^nz/dt^n$ expressed in terms of derivatives of z with regard to x, and of x with regard to t, and not replace x by its value in terms of t.

6. Consider next the following expansion problem. Given a function $F(a, g)$; replace a by $a + x$, and g by $g + y$, where

$$x = \alpha_1 h + (\alpha_2/2!)h^2 + (\alpha_3/3!)h^3 + \cdots,$$
$$y = \beta_1 h + (\beta_2/2!)h^2 + (\beta_3/3!)h^3 + \cdots.$$

Denote the result of the substitution by $F(h)$, and let it be required to expand $F(h)$ as a series in h.

Put

$$\mathfrak{D} = \partial/\partial a, \qquad \mathfrak{D}' = \partial/\partial g.$$

It is apparent that we can write symbolically

$$\begin{aligned} F(h) &= F(a + x, g + y) \\ &= [1 + h\mathcal{D}_1 + (h^2/2!)\,\mathcal{D}_2 + (h^3/3!)\,\mathcal{D}_3 + \cdots]\,F(a, g), \end{aligned}$$

where $\mathcal{D}_1, \mathcal{D}_2, \cdots$ are operators, which are polynomials in $\mathfrak{D}$ and $\mathfrak{D}'$, with the customary convention that $\mathfrak{D}^nF = \partial^nF/\partial a^n$, etc.

We are to find the operators $\mathcal{D}_1, \mathcal{D}_2, \cdots$.

We can write symbolically *

$$F(a + x, g + y) = (\exp[x\mathfrak{D} + y\mathfrak{D}'])F.$$

This leads us to put

$$\exp[x\mathfrak{D} + y\mathfrak{D}'] = E^z = G(h)$$

where

$$\begin{aligned} z = {} & [\alpha_1 h + (\alpha_2/2!)h^2 + \cdots]\,\mathfrak{D} \\ & + [\beta_1 h + (\beta_2/2!)h^2 + \cdots]\,\mathfrak{D}'. \end{aligned}$$

We have accordingly

$$G(h) = G(0) + h(dG/dh)_{h=0} + (h^2/2!)(d^2G/dh^2)_{h=0} + \cdots.$$

---

* In accordance with astronomical usage we let E denote the base of the system of natural logarithms.

The quantity d^nG/dh^n is the nth derivative of a function of z, where z is a function of h. We can therefore employ (5) to calculate the coefficients in the expansion of $G(h)$. It is seen that in the expression for z we are to manipulate with $\mathfrak{D}$ and $\mathfrak{D}'$ as with ordinary numbers. We can therefore write

$$\frac{dG}{dh} = E^z \frac{dz}{dh},$$

$$\frac{d^2G}{dh^2} = \frac{\partial}{\partial z}\left[\frac{dG}{dh}\,\frac{dz}{dh} + G\,\frac{d^2z}{dh^2}\right],$$

$$\frac{d^3G}{dh^3} = \frac{\partial}{\partial z}\left[\frac{d^2G}{dh^2}\,\frac{dz}{dh} + 2\,\frac{dG}{dh}\,\frac{d^2z}{dh^2} + G\,\frac{d^3z}{dh^3}\right],$$

A little attention to the expressions just given shows that the only way in which z enters the quantities within the signs [ ] is through the presence of E^z as a factor of dG/dh, d^2G/dh^2, $\cdots$. Since $z = 0$, when $h = 0$, and since

$$(d^nG/dh^n)_{h=0} = \mathcal{D}_n,$$

we obtain

$$\begin{aligned}
\mathcal{D}_1 &= (dz/dh)_{h=0},\\
\mathcal{D}_2 &= \mathcal{D}_1(dz/dh)_{h=0} + (d^2z/dh^2)_{h=0},\\
\mathcal{D}_3 &= \mathcal{D}_2(dz/dh)_{h=0} + 2\,\mathcal{D}_1(d^2z/dh^2)_{h=0} + (d^3z/dh^3)_{h=0},\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

But

$$\begin{aligned}
dz/dh &= (\alpha_1 + \alpha_2 h + \cdots)\mathfrak{D} + (\beta_1 + \beta_2 h + \cdots)\mathfrak{D}',\\
d^2z/dh^2 &= (\alpha_2 + \alpha_3 h + \cdots)\mathfrak{D} + (\beta_2 + \beta_3 h + \cdots)\mathfrak{D}',\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

so that we have finally

$$\begin{aligned}
\mathcal{D}_1 &= \alpha_1\mathfrak{D} + \beta_1\mathfrak{D}',\\
\mathcal{D}_2 &= (\alpha_1\mathfrak{D} + \beta_1\mathfrak{D}')\mathcal{D}_1 + (\alpha_2\mathfrak{D} + \beta_2\mathfrak{D}'),\\
\mathcal{D}_3 &= (\alpha_1\mathfrak{D} + \beta_1\mathfrak{D}')\mathcal{D}_2 + 2(\alpha_2\mathfrak{D} + \beta_2\mathfrak{D}')\quad {}_1 + (\alpha_3\mathfrak{D} + \beta_3\mathfrak{D}'),\\
\mathcal{D}_4 &= (\alpha\mathfrak{D}_1 + \beta_1\mathfrak{D}')\mathcal{D}_3 + 3(\alpha_2\mathfrak{D} + \beta_2\mathfrak{D})\,\mathcal{D}_2 + 3(\alpha_3\mathfrak{D} + \beta_3\mathfrak{D}')\,\mathcal{D}_1\\
&\quad + (\alpha_4\mathfrak{D} + \beta_4\mathfrak{D}'),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

It would be possible to express $\mathcal{D}_2$, $\mathcal{D}_3$, $\cdots$ explicitly in terms of $\mathfrak{D}$ and $\mathfrak{D}'$, but it would not be advantageous to do so.

7. We next define fundamental operators as follows:

$$
\begin{aligned}
&\theta_1 = \alpha_1\mathfrak{D} + \beta_1\mathfrak{D}',\\
(6)\qquad &\theta_2 = \alpha_2\mathfrak{D} + \beta_2\mathfrak{D}',\\
&\theta_3 = \alpha_3\mathfrak{D} + \beta_3\mathfrak{D}',\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
$$

We can then write:

$$
\begin{aligned}
&\mathcal{D}_1 = \theta_1,\\
&\mathcal{D}_2 = \theta_1\mathcal{D}_1 + \theta_2,\\
(7)\qquad &\mathcal{D}_3 = \theta_1\mathcal{D}_2 + 2\theta_2\mathcal{D}_1 + \theta_3,\\
&\mathcal{D}_4 = \theta_1\mathcal{D}_3 + 2\theta_2\mathcal{D}_2 + 3\theta_3\mathcal{D}_1 + \theta_4,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
$$

If it should happen that the function $F(a, g)$ on which we have to operate with $\mathcal{D}_1, \mathcal{D}_2, \cdots$ is such that the effect of operating on it with the fundamental operators $\theta_1\ \theta_2 \cdots$ is to merely replace a single term by a group of terms each with the general character of the original term, it is seen that the knowledge of the effect produced on F by $\theta_1, \theta_2, \cdots$ leads at once to the effect produced by $\mathcal{D}_1, \mathcal{D}_2, \cdots$. Such a condition is actually fulfilled in the case of the disturbing function, and it is due to this fact that the operators owe their power.

8. We shall proceed now to the application of the general method to the case of the disturbing function.

As stated above we consider the outer orbit to be circular and take for the function $F(a, g)$ of §§ 5-7 the function

$$N \cos \mu g,$$

of § 3, which represents a term of the expansion of Δ_0^{-1}, with the omission of symbols that are irrelevant for the moment.

We are to replace a by r, and g by the true anomaly f. Since we are considering N as a function of $\log a$, we must think of replacing $\log a$ by $\log r$. We can write

$$
\begin{aligned}
\log r &= \log a + x,\\
f &= g + y,
\end{aligned}
$$

and there are well known expansions for x and y in terms of the eccentricity e. We put

$$
\begin{aligned}
x &= e\alpha_1 + (e^2/2!)\alpha_2 + (e^3/3!)\alpha_3 + \cdots,\\
y &= e\beta_1 + (e^2/2!)\beta_2 + (e^3/3!)\beta_3 + \cdots.
\end{aligned}
$$

Instead of writing the α_i and the β_i in their usual trigonometric form, we shall introduce a convenient notation.

We put

$$(j)^+ = E^{jg\iota} + E^{-jg\iota},$$
$$(j)^- = E^{jg\iota} - E^{-jg\iota},$$

where $\iota = \sqrt{-1}.$

The following relations, easy to derive, and simple to employ, form the symbolic algebra for the development.

$$(8) \quad \begin{array}{ll} (1)^+(j)^+ = (j+1)^+ + (j-1)^+, & (1)^-(j)^- = (j+1)^+ - (j-1)^+, \\ (2)^+(j)^+ = (j+2)^+ + (j-2)^+, & (2)^-(j)^- = (j+2)^+ - (j-2)^+, \\ \cdots & \cdots \end{array}$$

In terms of the new symbol we have: *

$$(9) \quad \begin{array}{l} \alpha_1 = -(1/2)(1)^+, \qquad \alpha_2 = -(3/4)(2)^+ + (1/2), \\ \alpha_3 = -(17/8)(3)^+ + (9/8)(1)^+, \\ \alpha_4 = -(71/8)(4)^+ + (11/2)(2)^+ + (3/4), \\ \alpha_5 = -(1569/32)(5)^+ + (1155/32)(3)^+ + (15/16)(1)^+, \\ \cdots \end{array}$$

$$(10) \quad \begin{array}{l} \beta_1 = -\iota(1)^-, \qquad \beta_2 = -(5/4)\iota(2)^-, \\ \beta_3 = -(13/4)\iota(3)^- + (3/4)\iota(1)^-, \\ \beta_4 = -(103/8)\iota(4)^- + (11/2)\iota(2)^-, \\ \beta_5 = -(1097/16)\iota(5)^- + (645/16)\iota(3)^- - (25/8)\iota(1)^-, \end{array}$$

9. Since we are considering N as a function of log a, the operator $\mathfrak{D}$ of the general theory stands for ∂/∂ log a, that is, $\mathfrak{D}$ becomes the $\mathscr{D}$ of § 4. Therefore by (3) we have

$$(11) \qquad \mathfrak{D}N = \mathscr{D}N = DN,$$

where $D = \partial/\partial \log \alpha$. As stated before there are methods for computing the values of DN, D^2N, $\cdots$ for any pair of planets.

The operator $\mathfrak{D}'$ of the general theory remains $\partial/\partial g$. In view of this and the definition of $(j)^+$ and $(j)^-$, it is easy to see that

* The symbolic expressions can be obtained by obvious changes made on the expansions given by Newcomb, *loc. cit.*, Vol. 5, p. 21. It might be said that the value of α, which is the least common of the two expansions, can be obtained directly quite easily in the form used here, without using the method of development that Newcomb used.

$$\mathfrak{D}(j)^{+} = \iota j(j)^{-}. \tag{12}$$

10. We shall now develop the fundamental operators $\theta_1, \theta_2, \cdots$ of the present theory.

We first write

$$N \cos \mu g = N'(\mu)^{+}, \qquad N' = N/2.$$

Using (6), (9), and (10) we see that

$$\begin{aligned} \theta_1 N \cos \mu g &= \theta_1 N'(\mu)^{+} \\ &= [-(1/2)(1)^{+}\mathfrak{D} - \iota(1)^{-}\mathfrak{D}']N'(\mu)^{+}. \end{aligned}$$

Remembering that $\mathfrak{D}$ operates only on N', and using (11) and (12) we find

$$\theta_1 N'(\mu)^{+} = [-(1/2)(1)^{+}(\mu)^{+}D + \mu(1)^{-}(\mu)^{-}]N'.$$

When this is simplified by means of (8) we obtain

$$\begin{aligned} \theta_1 N'(\mu)^{+} = [-(1/2)\{(\mu+1)^{+} + (\mu-1)^{+}\}D& \\ + \mu\{(\mu+1)^{+} - (\mu-1)^{+}\}]N'.& \end{aligned}$$

Since only symbolic parentheses with the $+$ sign remain, we can drop the distinguishing mark.* Collecting similar terms we have finally

$$\begin{aligned} \theta_1 N'(\mu) &= [-(1/2)D + \mu]N' \cdot (\mu+1) \\ &\quad + [-(1/2)D - \mu]N' \cdot (\mu-1). \end{aligned}$$

For θ_2 we find

$$\begin{aligned} \theta_2 N'(\mu)^{+} &= [\{-(3/4)(2)^{+} + (1/2)\}(\mu)^{+}D + \mu(5/4)(2)^{-}(\mu)^{-}]N' \\ &= [\{-(3/4)(\mu+2) - (3/4)(\mu-2) + (1/2)(\mu)\}D \\ &\quad + \mu\{(5/4)(\mu+2) - (5/4)(\mu-2)\}]N' \end{aligned}$$

which for actual employment is rearranged below. The values of $\theta_3, \theta_4, \cdots$ are found similarly.

---

* Parentheses ( ) are symbolic except when enclosing a numerical fraction.

$$
\begin{aligned}
\theta_1 N'(\mu) &= [-(1/2)D + \mu]N' \cdot (\mu + 1) \\
&\quad + [-(1/2)D - \mu]N' \cdot (\mu - 1), \\
\theta_2 N'(\mu) &= [-(3/4)D + (5/4)\mu]N' \cdot (\mu + 2) + (1/2)DN' \cdot (\mu) \\
&\quad + [-(3/4)D - (5/4)\mu]N' \cdot (\mu - 2), \\
\theta_3 N'(\mu) &= [-(17/8)D + (13/4)\mu]N' \cdot (\mu + 3) \\
&\quad + [(9/8)D - (3/4)\mu]N' \cdot (\mu + 1) \\
&\quad + [(9/8)D + (3/4)\mu]N' \cdot (\mu - 1) \\
&\quad + [-(17/8)D - (13/4)\mu]N' \cdot (\mu - 1), \\
\theta_4 N'(\mu) &= [-(71/8)D + (103/8)\mu]N' \cdot (\mu + 4) \\
&\quad + [(11/2)D - (11/2)\mu]N' \cdot (\mu + 2) + (3/4)DN' \cdot (\mu) \\
&\quad + [(11/2)D + (11/2)\mu]N' \cdot (\mu - 2) \\
&\quad + [-(71/8)D - (103/8)N']N \cdot (\mu - 4), \\
\theta_5 N'(\mu) &= [-(1569/32)D + (1097/16)\mu]N' \cdot (\mu + 5) \\
&\quad + [(1155/32)D - (645/16)\mu]N' \cdot (\mu + 3) \\
&\quad + [(15/16)D + (25/8)\mu]N' \cdot (\mu + 1) \\
&\quad + [(15/16)D - (25/8)\mu]N' \cdot (\mu - 1) \\
&\quad + [(1155/32)D + (645/16)\mu]N' \cdot (\mu - 3) \\
&\quad + [-(1569/32)D - (1097/16)\mu]N' \cdot (\mu - 5) \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

12. We are now ready to develop the operators $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\cdots$. Using the relations (7) we have

$$\mathfrak{D}_1 N'(\mu) = \theta_1 N' \cdot (\mu),$$

$$
\begin{aligned}
\mathfrak{D}_2 N'(\mu) &= [-(1/2)D + \mu]^2 N' \cdot (\mu + 2) + [(1/4)D^2 - \mu^2]N' \cdot (\mu) \\
&\quad + [(1/4)D^2 - \mu^2]N' \cdot (\mu) + [-(1/2)D - \mu]^2 N' \cdot (\mu - 2) \\
&\quad + \theta_2,
\end{aligned}
$$

or using the value of θ_2 and collecting,

$$
\begin{aligned}
\mathfrak{D}_2 N'(\mu) &= [(1/4)D^2 + \{-\mu - (3/4)\}D + \{\mu^2 + (5/4)\mu\}]N' \cdot (\mu + 2) \\
&\quad + [(1/2)D^2 + (1/2)D - 2\mu^2]N' \cdot (\mu) \\
&\quad + [(1/4)D^2 + \{\mu - (3/4)\}D + \{\mu^2 - (5/4)\mu\}]N' \cdot (\mu - 2).
\end{aligned}
$$

For $\mathcal{T}_3$ we find

$$
\begin{aligned}
\mathfrak{D}_3 N'(\mu) &= [-(1/2)D + \mu][(1/4)D^2 + \{-\mu - (3/4)\}D \\
&\qquad\qquad + \{\mu^2 + (5/4)\mu\}]N' \cdot (\mu + 3) \\
&\quad + [-(1/2)D + \mu][(1/2)D^2 + (1/2)D - 2\mu^2]N' \cdot (\mu + 1)
\end{aligned}
$$

$$+ [-(1/2)D + \mu] [(1/4)D^2 + \{\mu - (3/4)D + \{\mu^2 - (5/4)\mu\}]N' \cdot (\mu - 1)$$

$$+ [-(1/2)D - \mu] [(1/4)D^2 + \{-\mu - (3/4)\}D + \{\mu^2 + (5/4)\mu\}]N' \cdot (\mu + 1)$$

$$+ [-(1/2)D - \mu] [(1/2)D^2 + (1/2)D - 2\mu^2]N' \cdot (\mu - 1)$$

$$+ [-(1/2)D - \mu] [(1/4)D^2 + \{\mu - (3/4)\}D + \{\mu^2 - (5/4)\mu\}]N' \cdot (\mu - 3)$$

$$+ 2[-(3/4)D + (5/4)\mu][-(1/2)D + \mu]N' \cdot (\mu + 3)$$

$$+ 2[-(3/4)D + (5/4)\mu][-(1/2)D - \mu]N' \cdot (\mu + 1)$$

$$+ 2 \cdot (1/2)D[-(1/2)D + \mu]N' \cdot (\mu + 1)$$

$$+ 2 \cdot (1/2)D[-(1/2)D - \mu]N' \cdot (\mu - 1)$$

$$+ 2[-(3/4)D - (5/4)\mu][-(1/2)D + \mu]N' \cdot (\mu - 1)$$

$$+ 2[-(3/4)D - (5/4)\mu][-(1/2)D - \mu]N' \cdot (\mu - 3)$$

$$+ \theta_3.$$

Inserting the value of θ_3 and collecting we obtain,

$$\begin{aligned}
\mathfrak{D}_3 N'(\mu) = {} & [-(1/8)D^3 + \{(3/4)\mu + (9/8)\}D^2 \\
& -\{(3/2)\mu_2 + (33/8)\mu - (17/8)\}D \\
& + \mu^3 + (15/4)\mu^2 + (13/4)\mu]N' \cdot (\mu + 3) \\
& + [-(3/8)D^3 + \{(3/4)\mu + (3/8)\}D_2 \\
& +\{(3/2)\mu^2 + (15/8)\mu + (9/8)\}D \\
& - 3\mu^3 - (15/4)\mu^2 - (3/4)\mu]N' \cdot (\mu + 1) \\
& + [-(3/8)D_3 + \{-(3/4)\mu + (3/8)\}D^2 \\
& + \{(3/2)\mu^2 - (15/8)\mu + (9/8)\}D \\
& + 3\mu^3 - (15/4)\mu^2 - (3/4)\mu]N' \cdot (\mu - 1) \\
& + [-(1/8)D^3 + \{-(3/4)\mu + (9/8)\}D^2 \\
& + \{-(3/2)\mu^2 + (33/8)\mu - (17/8)\}D \\
& - \mu^3 + (15/4)\mu^2 - (13/4)\mu]N' \cdot (\mu - 3).
\end{aligned}$$

The actual calculations will not be given further than this. They become long on account of the inherent complexity of the problem. It is seen, however, that the work consists almost entirely of the multiplication of polynomials.

13. When we return to trigonometric functions, it is seen that

$$\cdots N'(\mu + 3), \quad N'(\mu + 2), \cdots$$

become, respectively,

$$\cdots N \cos(\mu g + 3g), \quad N \cos(\mu g + 2g), \cdots$$

and that operational coefficients on N' will be applied unaltered to N.

It is also recalled that $\mathfrak{D}_s/s!$ is the coefficient of e^s in the series that replaces the term $N \cos \mu g$, when the eccentricity of the inner orbit is taken into account.

We can summarize the results as follows:

When account is taken of the eccentricity e, the single term $N \cos \mu g$ of Δ_0^{-1} goes into an infinite series with terms of the form

$$\Pi_j^s \, N e^s \cos(\mu g + jg),$$

where Π_j^s is an operational polynomial in D, the derivative with regard to $\log \alpha$. The letter s takes the values $0, 1, 2, \cdots$, while for a given s, the letter j takes the restricted set of values

$$s, \quad s-2, \quad s-4, \cdots, \quad -s+2, \quad -s.$$

The operators Π_j^s are the *operators of Newcomb.*

It is seen that our operator $\mathfrak{D}_s$ gives a group of Newcomb operators. For instance, the operator $\mathfrak{D}_3$ gives Π_3^3, Π_1^3, Π_{-1}^3, Π_{-3}^3. It is to be observed that account must be taken of the factor $1/s!$ in passing from the operators to the operators Π.

A few of the operators of Newcomb are given below; they can all be obtained from the expressions previously given.*

$$\begin{aligned}
\Pi_0^0 &= 1, \\
2\Pi_1^1 &= 2\mu - D, \\
8\Pi_2^2 &= 4\mu^2 + 5\mu + (-4\mu - 3)D + D^2, \\
4\Pi_0^2 &= -4\mu^2 + D + D^2, \\
48\Pi_3^3 &= 8\mu^3 + 30\mu^2 + 26\mu + (-12\mu^2 - 33\mu - 17)D \\
&\quad + (6\mu + 9)D^2 - D^3, \\
16\Pi_1^3 &= -8\mu^3 - 10\mu^2 - 2\mu + (4\mu^2 + 5\mu + 3)D \\
&\quad + (2\mu + 1)D^2 - D^3.
\end{aligned}$$

The operators must, of course, be evaluated separately for terms of class $0, 1, 2, \cdots$. (See § 3). Thus for terms of class 0 we put $\mu = -i$. If this

* Parentheses ( ) are no longer symbolic.

be done in the operators above, we shall find that the results agree with those given by Newcomb.*

14. For the sake of completeness we shall show how to take into account the eccentricity e' of the outer orbit.

Since introducing the eccentricity e results in replacing every term of Δ_0^{-1} by a series, each term of which is of the same form as the term it replaces, and since the introduction of e' is, from the standpoint of the analytic processes involved, precisely the same as that of introducing e, it is seen we have merely to apply a second time the processes already developed.

Consider then the term $N\cos(\mu g + \nu g')$. When e and e' are both taken into account, this term will give rise to a series whose general term can be written

$$\Pi_j{}^s\,(\Pi_{j'}{}^{s'})\;N\cdot e^s\,e'^{s'}\cos(\mu g + \nu g' + jg + j'g'),$$

where $(\Pi_{j'}{}^{s'})$ is obtained from $\Pi_j{}^s$ by replacing D by $\mathfrak{D}' = \partial/\partial \log a'$, and μ by ν.

The quantities s and s' take the values $0, 1, 2, \cdots$, independently of each other, while for a fixed s and s',

$$\begin{aligned} j &= s,\ s-2,\ \cdots,\ -s+2,\ -s,\\ j' &= s',\ s'-2,\ \cdots,\ -s'+2,\ -s'. \end{aligned}$$

The operator $\mathfrak{D}'$ can be entirely eliminated, for we have by (4), $\mathfrak{D} = -1 - D$.

For the operator $(\Pi_{j'}{}^{s'})$ when expressed in terms of D, Newcomb uses the symbol $\Pi^{0,\,s'}_{0,\,j'}$. We have the rule:

To obtain $\Pi^{0,\,s'}_{0,\,j'}$ replace D in $\Pi_{j'}{}^{s'}$ by $-1-D$, and μ by ν.

Some of the new operators are:

$$\begin{aligned} \Pi^{0,1}_{0,1} &= 1,\\ 2\Pi^{0,1}_{0,1} &= 2\nu + 1 + D,\\ 8\Pi^{0,2}_{0,2} &= 4\nu^2 + 9\nu + 4 + (4\nu + 5)D + D^2,\\ 4\Pi^{0,2}_{0,0} &= -4\nu^2 + D + D^2. \end{aligned}$$

Again, it is necessary to give to ν the value appropriate to the class of terms considered before comparing with the results of Newcomb.

It will merely be remarked that it is natural to combine the two operators by writing

$$\Pi^{s,\,s'}_{j,\,j'} = \Pi_j{}^s\,\Pi^{0,\,s'}_{0,\,j'},$$

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* *Loc. cit.*, Vol. 5, pp. 27 ff.

but it is unnecessary to go into the details. The process is apparent and the results are given by Newcomb.

15. Throughout the work the argument of the angle has been incompletely represented by the omission of ω and ω'. It is easy to see how they should be restored. For terms of class 0, 1, 2, $\cdots$, we replace

$$\cos(\mu g + \nu g + jg + j'g')$$

by

$$\cos(V_i + jg + j'g'),$$
$$\cos(V_i^{(1)} + jg + j'g'),$$
$$\cdots\cdots\cdots$$

respectively, where

$$V_i = i\lambda' - i\lambda = i(\omega' + g') - i(\omega + g),$$
$$V_i^{(1)} = (i+1)(\omega' + g') - (i-1)(\omega + g),$$
$$\cdots\cdots\cdots$$

The results can then be put in a form suitable for computation.

16. It is seen that the key to the development which has been given here lies in the fundamental operators, and the employment of equations (7), in which the coefficients are the binomial coefficients. There are only a small number of the fundamental operators and relations, and they are all comparatively simple. In Newcomb's work the existence of a group of fundamental operators does not appear to be recognized. The symbolic parentheses are of course a minor detail. The usual trigonometric functions could have been used. Yet the notation for the latter is at times unfortunately cumbersome. In a problem so inherently complex as that considered here, it seems desirable to adopt every possible convenience of writing. It appears to the writer that the symbols used contribute sensibly to ease of manipulation and to brevity.

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An Extension of Maschke's Symbolism.*

By Nola Lee Anderson.

1. *Introduction.* This paper considers a generalization of the symbolic method of Maschke for representing the invariants of a quadratic differential form $\sum g_{ij}dx^i dx^j$. According to this method, the coefficients g_{ij} are represented symbolically as products of $f_i f_j$.

Maschke regarded f_i and f_j as partial derivatives, $\partial f/\partial x^i$, $\partial f/\partial x^j$, of a symbolic function f, and placed

$$(1) \qquad A = \sum g_{ij}dx^i dx^j = (df)^2 = (\sum f_i dx^i)^2.$$

The possibility of a similar symbolic representation in which it is not assumed that $\partial f_i/\partial x^j = \partial f_j/\partial x^i$ has been suggested by Wilson and Moore.† This is the generalization that is to be considered in this paper.

As might be expected this generalization does not affect the mode of representing first order differential parameters. It is found, however, that invariant forms of higher order can best be regarded as invariants of the set of quantities g_{ij} together with certain other sets.

The two dimensional case is discussed in some detail. A geometric interpretation is introduced,‡ and certain invariant vectors related to a surface are studied.

The $f_1(u, v)$ and $f_2(u, v)$ are here regarded as any pair of independent vectors, satisfying the relation (1), associated with the point u, v of the surface. We find that the plane (called the base plane) which they determine has many properties analogous to properties of the tangent plane in the ordinary case. The effect of varying the orientation of the vectors f_1 and f_2 in the base plane is considered only in a special case.

Normals to this plane which are expressible in terms of the first deriva-

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* Presented to the American Mathematical Society, November 27, 1926.

† "Differential Geometry of Two Dimensional Surface in Hyperspace," *Proceedings of the American Academy of Arts and Sciences*, Vol. 52 (1916), p. 294.

‡ A geometric interpretation of the Symbolic Method of Maschke was made and studied in detail by L. Ingold in the paper "A Symbolic Treatment of the Geometry of Hyperspace," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 574-599.

tives of f_1 and f_2 are called the first normals relative to the base plane. In general, there are four independent first normals to the base plane instead of three as in the usual case.*

The analogues of the various curvature vectors associated with the curves $a(u, v) =$ constant are studied. These vectors in the main have properties very similar to the properties of the usual curvature vectors. There are, however, certain novel features. In particular we mention the appearance of a new invariant vector.

2. *First order invariants.* In order to represent expressions of higher than the first degree in g_{ij} by means of symbols, Maschke introduced symbols ϕ_i, ψ_i, etc. equivalent to the f_i. Thus, $g_{ij}g_{rs}$ might be written symbolically as $f_i f_j \cdot \phi_r \phi_s$, and so on.

The fundamental theorem states that if $A^1, \cdots, A^n$ are any n invariants of the quadratic differential form $G \equiv \sum g_{ij}dx^i dx^j$ in n variables, then β times their Jacobian is also an invariant where β is the reciprocal of the square root of the discriminant of G. This is written in the abbreviated form $(A^1, A^2, \cdots, A^n)$, where the factor β is understood.

In the usual sense, arbitrary functions are invariants so that some or all of the A's may stand for arbitrary functions of $x^1, \cdots, x^n$. This applies to the symbolic functions f, and, likewise, some or all of the A's may stand for any of the equivalent symbolic functions f, ϕ, ψ, etc. Again, the product of any number of the invariants is invariant. This fact enables us to construct invariants involving the g_{ij} by writing products in which the symbol f occurs in two distinct factors; therefore $(f A^1 \cdots A^{n-1})(f B^1 \cdots B^{n-1})$ is an invariant containing the quantities g_{ij}.

In generalizing these ideas, we shall continue to use the symbolism of Maschke and let $f_i f_j = g_{ij}$, where f_i and f_j, however, are not regarded as partial derivatives. It becomes necessary, then, to give their law of transformation. It has been noticed that the argument by which it is shown that $(A^1, \cdots, A^r)$ is an invariant is based chiefly on the fact that the derivatives $\partial A^i/\partial x^j$ transform according to the law

$$\partial \bar{A}^i/\partial \bar{x}^s = \sum (\partial A^i/\partial x^j)(\partial x^j/\partial \bar{x}^s)$$

when the variables are changed from $x^1, \cdots, x^n$ to $\bar{x}^1, \cdots, \bar{x}^n$. In accordance with this, we assume that symbols f_i follow the same law of transformation

$$\bar{f}_r = \sum f_j (\partial x^j/\partial \bar{x}^r) \tag{2}$$

with similar equations for equivalent symbols.

* See Wilson and Moore, *loc. cit.*, p. 304.

It is evident that the invariant expressions used by Maschke are still formally invariant with this extended conception of the f_i, ϕ_i, etc. Throughout this discussion, it will be understood that parenthesis expressions containing the symbolic functions have the same meaning as in Maschke's theory except that f, ϕ, etc. represent columns $f_1, f_2, \cdots, f_n, \phi_1, \phi_2, \cdots, \phi_n$, etc. instead of derivatives. Some of the elements of the parenthesis expressions may be parenthesis expressions themselves, as $((fa)a)(fb)$. In such cases, $(fa)_1$, means $\partial(fa)/\partial x^1$, etc., and, in general, unless the contrary is specified, the elements of a column of any parenthesis will be understood to be the partial derivatives of the quantity at the head of the column excepting only the case of f or its equivalent.

It is seen immediately that any symbolic invariant expression represents exactly the same invariant whether the f_i are regarded as derivatives or not, *provided no derivatives of the* f_i, ϕ_i, *etc. appear.*

3. *Triple index symbols.* The invariant expressions involving first derivatives of the fundamental quantities or of arbitrary functions are called invariants of the first order. It is clear that the generalization proposed in this paper does not affect the representation of such invariants. But other invariant expressions may occur which contain derivatives of the f_i, that is, such quantities as f_{ij}, where f_{ij} denotes the partial derivative, $\partial f^i/\partial x^j$.* Then, invariants of this type are called invariants of the second order.

From the equations $f_j f_j = g_{jj}$, we find by differentiation with respect to x^i that

(3) $$f_{ji} f_j = (1/2)\partial g_{jj}/\partial x^i.$$

The products $f_i f_{jk}$ will be denoted by $[jk, i]$; these are analogous to the usual Christoffel triple index symbols.

If the equation $f_i f_j = g_{ij}$, where $i \neq j$, is differentiated with respect to x^k, we obtain

(4) $$f_i f_{jk} + f_j f_{ik} = \partial g_{ij}/\partial x^k.$$

In the usual treatment the expressions $[ij, k]$ and $[ji, k]$ are the same so that the equations (3) and (4) are sufficient to determine all of the triple index symbols in terms of the derivatives of the fundamental quantities.

Here, however, this is no longer the case. But if notations, P, Q, etc., are introduced for a sufficient number of these symbols, all may be expressed in terms of the g_{ij} and these additional quantities,

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* Whenever double subscripts occur with a symbolic quantity, the second one always denotes differentiation.

$$P, Q, \cdots.$$

Furthermore, it is possible to express several combinations of $f_{ij}f_{km}$ and $f_i f_{jkm}$ in terms of the quantities g_{ij} together with the additional quantities just mentioned.

4. *Geometric interpretation.* A geometric interpretation of this symbolism for the two dimensional case will now be considered. Throughout the remainder of this paper, we shall let u and v denote the curvilinear coordinates of a two dimensional surface in space.* We shall also from this point on consider

$$g_{11} = E, \quad g_{12} = F, \quad g_{22} = G,$$

except in cases where it is advantageous to use the coefficients g_{ij} in the formulas.

As has been referred to previously, Maschke used the symbol f_i as $\partial f/\partial x^i$. To interpret this geometrically, $f(u, v)$ may be regarded as a vector from some arbitrary origin to points of a two dimensional surface in space. Hence, at each point, f_1 and f_2 will be considered as a pair of vectors which are the partial derivatives of f, with respect to u and v, and therefore are tangent to the parameter curves, $v =$ constant and $u =$ constant, respectively. Thus, f_1 and f_2 lie in the tangent plane to the surface. In this case, of course, $f_{12} = f_{21}$.

Under the proposed generalization, the $f_1(u, v)$ and $f_2(u, v)$ are interpreted as any two vectors associated with the point u, v of the surface subject to the conditions that $f_1^2 = E$, $f_1 f_2 = F$, and $f_2^2 = G$. The products $f_i f_j$ will be regarded as scalar products. The plane which these vectors determine will be called the base plane † of the corresponding point of the surface, and the vectors themselves will occasionally be referred to as the fundamental vectors or base vectors. The base plane and the fundamental vectors $\mathbf{f}_1$, $\mathbf{f}_2$ ‡ will be called the basis $[\mathbf{f}_1, \mathbf{f}_2]$.

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* This space is of unspecified number of dimensions.

† The introduction of an arbitrary base plane at each point of a surface in the place of the tangent plane is very similar to the introduction of an arbitrary line at each point of a curve instead of the tangent line. The effect of replacing the tangent vector to a curve by an arbitrary vector function of the arc length or other parameter was considered by W. Fr. Meyer in the paper, "Ausdehnung der Frenetschen Formeln und Verwandter auf den R_n," *Jahresbericht der Deutschen Mathematiker Vereinigung*, Vol. 19 (1910), pp. 160-169.

A similar method has been recently employed in the study of Riemannian geometry. See Eisenhart, *Riemannian Geometry*, p. 73 and p. 107 where further references will be found.

‡ From this point on bold-faced type will be used to indicate vectors.

It should be mentioned that the formulas of the theory will depend not only on the base plane but on the orientation of the fundamental vectors as well. The study of properties independent of this orientation is not included.

Since in the general case the restriction that $f_{12} = f_{21}$ is removed, the quantity f_{12} is considered as independent of the quantity f_{21}.

If in a special case the f_1, f_2 are the derivatives of a vector f, they are regarded as tangents to the parameter curves of the surface, and the base plane becomes the tangent plane. This case will be known as the ordinary or usual case. It is clear, however, that when the base plane coincides with the tangent plane, the fundamental vectors f_1, f_2 need not coincide with the tangents to the parameter curves. This will be illustrated in the special case which is considered in the next articles.

5. *A special case.* We consider here the case in which the base plane coincides with the tangent plane. There are first developed a number of general formulas which will be valuable later for purposes of reduction.

From the scalar products,

$$f_1 f_1 = E, \quad f_1 f_2 = F, \quad f_2 f_2 = G, \tag{5}$$

we deduce by differentiation the following equations for the triple index symbols:

$$\begin{aligned} &[11,1] = (1/2)E_1, \quad [21,2] = (1/2)G_1, \quad [21,1] + [11,2] = F_1, \\ &[12,1] = (1/2)E_2, \quad [22,2] = (1/2)G_2, \quad [12,2] + [22,1] = F_2. \end{aligned} \tag{6}$$

It is convenient now to introduce the additional quantities, P, Q, etc. mentioned in section 3. Therefore, we shall let $[11,2] = P$ and $[22,1] = Q$.

The Christoffel triple index symbols of the second kind will be used throughout this paper and are defined in the usual manner by the equations

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \sum g^{km} \, [ij, m]. \tag{7}$$

That is,

$$\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = g^{11}[12,1] + g^{12}[12,2], \text{ etc., where}$$

$$g^{11} = G\beta^2, \quad g^{12} = -F\beta^2, \quad g^{21} = -F\beta^2, \quad g^{22} = E\beta^2. \tag{8}$$

We may, also, express the triple index symbols of the first kind in terms of the triple index symbols of the second kind,

$$[ij, k] = \sum g_{km} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\}, \tag{9}$$

and by this relation we have equations of the type

$$[11,1] = g_{11} \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} + g_{12} \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} .$$

By taking the second derivatives of $f_1^2 = E$, $f_2^2 = G$, and $f_1 f_2 = F$, several combinations of $f_{ik} f_{lm}$ and $f_i f_{klm}$ can be expressed in terms of E, F, G, P, Q, and their derivatives, such as

$$f_{21} f_{11} + f_2 f_{111} = P_1, \; f_{11} f_{22} - f_{12} f_{21} = P_2 + Q_1 - F_{12}, \text{ etc.}$$

Now if the base plane coincides with the tangent plane the vectors f_1 and f_2 must be tangent vectors; they are therefore linearly expressible in terms of the vectors tangent to the parameter curves.

In the remaining portion of this section and in section 7 the notations φ_1 and φ_2 will be used to denote vectors tangent to the parameter curves $v =$ constant and $u =$ constant, respectively, subject to the conditions

$$\varphi_1^2 = E, \qquad \varphi_1 \varphi_2 = F, \quad \varphi_2^2 = G.$$

It is clear from these equations and equations (5) that the cosine of the angle between the vectors φ_1 and φ_2 is the same as the cosine of the angle between f_1 and f_2, and the angles themselves are either equal or supplementary. We consider the case in which they are equal.

Let $\theta(u, v)$ denote the angle between f_1 and φ_1 which is also the angle between f_2 and φ_2. We then have the formulas

$$f_1 \varphi_1 = E \cos \theta, \qquad f_2 \varphi_2 = G \cos \theta,$$

and therefore

$$(f_1 \varphi_1)/E = (f_2 \varphi_2)/G = \cos \theta.$$

These equations enable us to determine the coefficients p, q, m, n in the following equations

(10) $$f_1 = p\varphi_1 + q\varphi_2,$$

(11) $$f_2 = m\varphi_1 + n\varphi_2.$$

If (10) be multiplied by φ_1, and (11) by φ_2, the equations

$$p = \cos \theta - q(F/E)$$

and $$n = \cos \theta - m(F/G)$$

are obtained.

If (10) be squared, we find after substituting the value of p, that $q = (E \sin \theta)/(EG - F^2)^{1/2}$; * consequently,

$$p = \cos \theta - (F \sin \theta)/(EG - F^2)^{1/2}.$$

* The use of the opposite sign for q is equivalent to reversing the angle θ.

Again, when (10) is multiplied by (11) and the substitutions for p, q, and m are made, we have after simplifying, $m = (-G\sin\theta)/(EG-F^2)^{1/2}$, and consequently $n = \cos\theta + (F\sin\theta)/(EG-F^2)^{1/2}$.

Equations (10) and (11) may now be written

$$f_1 = (\cos\theta - \beta F\sin\theta)\varphi_1 + (\beta E\sin\theta)\varphi_2,$$
$$f_2 = -(\beta G\sin\theta)\varphi_1 + (\cos\theta + \beta F\sin\theta)\varphi_2$$
$$\text{where } \beta = 1/(EG-F^2)^{1/2}.$$

From these we can compute the derivatives of the base vectors, f_1 and f_2, with respect to u and v,

$$f_{11} = [\cos\theta - \beta F\sin\theta]\varphi_{11} + [\beta E\sin\theta]\varphi_{21} + [\cos\theta - \beta F\sin\theta]_1\varphi_1 + [\beta E\sin\theta]_1\varphi_2;$$
$$f_{12} = [\cos\theta - \beta F\sin\theta]\varphi_{12} + [\beta E\sin\theta]\varphi_{22} + [\cos\theta - \beta F\sin\theta]_2\varphi_1 + [\beta E\sin\theta]_2\varphi_2;$$
$$f_{21} = -[\beta G\sin\theta]\varphi_{11} + [\cos\theta + \beta F\sin\theta]\varphi_{21} - [\beta G\sin\theta]_1\varphi_1 + [\cos\theta + \beta F\sin\theta]_1\varphi_2;$$
$$f_{22} = -[\beta G\sin\theta]\varphi_{12} + [\cos\theta + \beta F\sin\theta]\varphi_{22} - [\beta G\sin\theta]_2\varphi_1 + [\cos\theta + \beta F\sin\theta]_2\varphi_2.$$

If the scalar product of f_2 and f_{11} is computed, $P = [11, 2]$ is obtained in terms of E, F, G, and θ. In a similar manner, $Q = [22, 1]$ can be found, and thus by equations (6) all of the expressions $[ij, k]$ are known.

6. *Normal vectors.* By normal vectors we mean vectors that are orthogonal to the fundamental system of vectors, that is, in this case vectors orthogonal to the base plane. The vectors f_{ij} are in general oblique to the base plane, but there exist normal vectors expressible in terms of the fundamental vectors f_1, f_2, and the vectors f_{11}, f_{12}, f_{21}, f_{22}. Such vectors will be called first normals. Since f_{11}, f_{12}, f_{21}, f_{22} are as a rule independent, there are in general four independent first normals. This paper is for the most part concerned only with those cases in which these are independent.

One such set of four independent normals is given by

$$N_{ij} = f_{ij} - \Sigma \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} f_m. \tag{12}$$

It is shown easily that N_{ji} and f_k are orthogonal, i. e. $N_{ij}f_k = 0$. If (12) is multiplied by f_k,

$$f_k N_{ij} = [ij, k] - \Sigma\, g_{km} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\}.$$

Hence, it follows immediately from (9) that

$$\boldsymbol{f}_k \boldsymbol{N}_{ij} = 0.$$

There are, of course, other sets of independent first normals. In some cases the following set is more convenient,

$$\text{(13)} \qquad \boldsymbol{a} = \phi_1(\boldsymbol{f}\phi)_1, \quad \boldsymbol{b} = \phi_2(\boldsymbol{f}\phi)_2, \quad \boldsymbol{c} = \phi_1(\boldsymbol{f}\phi)_2, \quad \boldsymbol{d} = \phi_2(\boldsymbol{f}\phi)_1.^*$$

It is readily proved, for example, that $\phi_1(\boldsymbol{f}\phi)_1$ is a normal vector.

$$(\boldsymbol{f}\phi)_1 = \beta_1[\boldsymbol{f}_1\phi_2 - \boldsymbol{f}_2\phi_1] + \beta[\boldsymbol{f}_{11}\phi_2 - \boldsymbol{f}_{21}\phi_1 + \boldsymbol{f}_1\phi_{21} - \boldsymbol{f}_2\phi_{11}],$$

and if this be multiplied by ϕ_1,

$$\phi_1(\boldsymbol{f}\phi)_1 = \beta_1[F\boldsymbol{f}_1 - E\boldsymbol{f}_2] + \beta[F\boldsymbol{f}_{11} - E\boldsymbol{f}_{21} + [21,1]\boldsymbol{f}_1 - [11,1]\boldsymbol{f}_2].$$

Simplifying and substituting the values of $\boldsymbol{f}_{11}$ and $\boldsymbol{f}_{21}$ from (12),

$$\phi_1(\boldsymbol{f}\phi)_1 = \{\beta_1 F + \beta[21,1]\}\boldsymbol{f}_1 - \{\beta_1 E_2 + \beta[11,1]\}\boldsymbol{f}_2$$

$$+ \beta F\left(\boldsymbol{N}_{11} + \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} \boldsymbol{f}_1 + \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} \boldsymbol{f}_2\right) - \beta E\left(\boldsymbol{N}_{21} + \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \boldsymbol{f}_1 + \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} \boldsymbol{f}_2\right)$$

$$= \left(\beta_1 F + \beta[21,1] + \beta F \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \beta E \begin{Bmatrix} 1 \\ 21 \end{Bmatrix}\right)\boldsymbol{f}_1$$

$$- \left(\beta_1 E + \beta[11,1] - \beta F \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} + \beta E \begin{Bmatrix} 2 \\ 21 \end{Bmatrix}\right)\boldsymbol{f}_2 + \beta F\boldsymbol{N}_{11} - \beta E\boldsymbol{N}_{21}.$$

Since $\beta = 1/(EG - F^2)^{1/2}$,

$$\beta_1 = -(E_1 G + EG_1 - 2FF_1)/2(EG - F^2)^{3/2}$$
$$= -\beta^3[(1/2)E_1 G + (1/2)G_1 E - FF_1],$$

and by (6),

$$\beta_1 = -\beta^3 G[11,1] - \beta^3 E[21,2] + \beta^3 F[21,1] + \beta^3 F[11,2].$$

Now by substituting this value β_1 in the last equation for $\phi_1(\boldsymbol{f}\phi)_1$, we find

$$\boldsymbol{a} = \phi_1(\boldsymbol{f}\phi)_1 = \beta(F\boldsymbol{N}_{11} - E\boldsymbol{N}_{21}).$$

In like manner,

$$\text{(14)} \qquad \begin{aligned} \boldsymbol{b} &= \phi_2(\boldsymbol{f}\phi)_2 = \beta(G\boldsymbol{N}_{12} - F\boldsymbol{N}_{22}), \\ \boldsymbol{c} &= \phi_1(\boldsymbol{f}\phi)_2 = \beta(F\boldsymbol{N}_{12} - E\boldsymbol{N}_{22}), \\ \boldsymbol{d} &= \phi_2(\boldsymbol{f}\phi)_1 = \beta(G\boldsymbol{N}_{11} - F\boldsymbol{N}_{21}). \end{aligned}$$

It is obvious that these are normal vectors.

---

* $(\boldsymbol{f}\phi)_1$, and $(\boldsymbol{f}\phi)_2$ denote the partial derivatives $\partial(\boldsymbol{f}\phi)/\partial u$ and $\partial(\boldsymbol{f}\phi)/\partial v$. It seems preferable in formulas of this kind to print only the notation $\boldsymbol{f}$, which appears just once, in bold-faced type to indicate the vectorial character of the expression.

We can now show without difficulty that the invariant vectors

$$(15) \qquad (\phi a)((f\phi)a), \quad (\phi a)(\psi a)((f\phi)\psi), \quad (\psi\phi)(\psi a)((f\phi)a),$$

where a denotes a function of u and v, are normal vectors since it is possible to express them linearly in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.

For example,

$$\begin{aligned}(\phi a)((f\phi)a) &= \beta[\phi_1 a_2 - \phi_2 a_1]\beta[(f\phi)_1 a_2 - (f\phi)_2 a_1] \\ &= \beta^2[\phi_1(f\phi)_1 a_2^2 - \phi_1(f\phi)_2 a_1 a_2 - \phi_2(f\phi)_1 a_1 a_2 \\ &\qquad - \phi_2(f\phi)_2 a_1^2 \\ &= a_2^2\boldsymbol{a} - a_1 a_2(\boldsymbol{c} - \boldsymbol{d}) + a_1^2\boldsymbol{b}.\end{aligned}$$

By expanding in a similar manner it is found that the invariant vector $(\psi\phi)(\psi(f\phi))$, is also a normal vector. (In the ordinary case this vector vanishes identically.)*

$$\begin{aligned}\text{Thus,}\quad (\psi\phi)(\psi(f\phi)) &= \beta^2\{[\psi_1\phi_2 - \psi_2\phi_1]\,[\psi_1(f\phi)_2 - \psi_2(f\phi)_1]\} \\ &= \beta^2(E\boldsymbol{b} - F\boldsymbol{d} - F\boldsymbol{c} + G\boldsymbol{a}).\end{aligned}$$

If the values of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ from equations (14) are substituted in this, we obtain

$$(16) \qquad (\psi\phi)(\psi(f\phi)) = \beta[\boldsymbol{N}_{12} - \boldsymbol{N}_{21}] = \boldsymbol{j}.$$

There is another invariant normal vector,

$$((f\phi)\phi) = \beta[(f\phi)_1\phi_2 - (f\phi)_2\phi_1], \text{ or by (13),}$$

$$(17) \qquad ((f\phi)\phi) = \beta(\boldsymbol{d} - \boldsymbol{c}) = \boldsymbol{h}^*.$$

It should be observed that the vectors $\boldsymbol{h}^*$ and $\boldsymbol{j}$ do not depend on an arbitrary function a.

7. *Application to special case.* It will be recalled that in the discussion of section 5 the base plane was made to coincide with the tangent plane to the surface and the fundamental vectors f_1, f_2 to differ from the tangent vectors φ_1, φ_2 only by a rotation through the angle θ. In this special case, it is possible to express the invariant normal vectors $\boldsymbol{j}$ and $\boldsymbol{h}^*$ in terms of θ and $\boldsymbol{h}$, where $\boldsymbol{h} = \beta(\boldsymbol{d} - \boldsymbol{c})$ in the ordinary case.

First, let us expand $\boldsymbol{h}$, regarding the φ_1 and φ_2 as tangent vectors, in the usual case.

---

* See L. Ingold, *loc. cit.*, p. 580.

$$\begin{aligned}\boldsymbol{h} = \beta(\boldsymbol{d} - \boldsymbol{c}) &= \beta[(\varphi\psi)_1\psi_2 - (\varphi\psi)_2\psi_2] \\ &= \beta\{\beta_1[\varphi_1\psi_2 - \varphi_2\psi_1]\psi_2 - \beta_2[\varphi_1\psi_2 - \varphi_2\psi_1]\psi_1 \\ &\quad + \beta\psi_2[\varphi_{11}\psi_2 - \varphi_{21}\psi_1 + \varphi_1\psi_{21} - \varphi_2\psi_{11}] \\ &\quad - \beta\psi_1[\varphi_{12}\psi_2 - \varphi_{22}\psi_1 + \varphi_1\psi_{22} - \varphi_2\psi_{12}]\},\end{aligned}$$

where ψ_1 ψ_2 are equivalent to φ_1, φ_2.
Simplifying and collecting,

$$\begin{aligned}\boldsymbol{h} = &\beta\beta_1(G\varphi_1 - F\varphi_2) - \beta\beta_2(F\varphi_1 - E\varphi_2) \\ &+ \beta^2(G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}) \\ &+ \beta^2\varphi_1([21,2] + [22,1]) - \beta^2\varphi_2([11,2] + [12,1]),\end{aligned}$$

where we make use of the fact $\varphi_{12} = \varphi_{21}$.

Thus, $\boldsymbol{h} = \beta^2[G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}]$ + terms in φ_1 and φ_2.

Next, we shall consider the invariant normal vector $\boldsymbol{j} = (\psi\phi)(\psi(\boldsymbol{f}\phi)) = \beta[\boldsymbol{N}_{12} - \boldsymbol{N}_{21}]$. When the values for $\boldsymbol{N}_{12}$ and $\boldsymbol{N}_{21}$ from (12) are substituted in this equation,

$$\boldsymbol{j} = \beta[\boldsymbol{f}_{12} - \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} \boldsymbol{f}_1 - \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \boldsymbol{f}_2 - \boldsymbol{f}_{21} + \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \boldsymbol{f}_1 + \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} \boldsymbol{f}_2].$$

Therefore,

$$\boldsymbol{j} = \beta[\boldsymbol{f}_{12} - \boldsymbol{f}_{21}] + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2.$$

From the values for $\boldsymbol{f}_{12}$ and $\boldsymbol{f}_{21}$ obtained in section 5,

$$\boldsymbol{f}_{12} - \boldsymbol{f}_{21} = \beta \sin\theta[G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}] + \text{terms in } \varphi_1 \text{ and } \varphi_2,$$

where φ_1 and φ_2 denote the tangent vectors to the parameter curves. It will, also, be recalled that $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ can be expressed in terms of φ_1 and φ_2. As a result of this, $\boldsymbol{j}$ can be written

$$\boldsymbol{j} = (\sin\theta)\boldsymbol{h} + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2.$$

Since $\boldsymbol{j}$ and $\boldsymbol{h}$ are normal vectors, the terms involving the tangent vectors, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, will vanish. Hence,

$$\boldsymbol{j} = (\sin\theta)\boldsymbol{h}. \tag{18}$$

Lastly, we shall consider the normal vector $\boldsymbol{h}^* = ((\boldsymbol{f}\boldsymbol{f}')\boldsymbol{f}')$ with $\boldsymbol{f}'$ equivalent to $\boldsymbol{f}$. When it is expanded in a manner similar to the expansion of $\boldsymbol{j}$,

$$\boldsymbol{h}^* = \beta^2 \cos\theta\,\{G\varphi_{11} - 2F\varphi_{12} + E\varphi_{22}\} + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2,$$

which may be written

$$\boldsymbol{h}^* = (\cos\theta)\boldsymbol{h} + \text{terms in } \boldsymbol{f}_1 \text{ and } \boldsymbol{f}_2.$$

By the same reasoning as in the above case,

$$h^* = (\cos\theta)h. \tag{19}$$

8. *Polar directions.* We shall now consider the curves $a =$ constant on the two dimensional surface whose coördinates are u and v.

If the quantity V is a scalar function of u and v, the invariant $(Va)/[(fa)^2]^{1/2}$, is the same as it was before this generalization was made. That is, $(Va)/[(fa)^2]^{1/2} = dV/ds$, which is the derivative of the scalar quantity V with respect to arc length s along the curves $a =$ constant.*

But this is not true with $(\boldsymbol{f}a)/[(fa)^2]^{1/2}$, which will be denoted by $\boldsymbol{t}$, because $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are not $\partial\boldsymbol{f}/\partial u$ and $\partial\boldsymbol{f}/\partial v$. This, however, is an invariant vector lying in the base plane. Any scalar multiple of this vector $(\boldsymbol{f}a)$ will be called a polar vector of the curve $a =$ constant with respect to the basis. The direction of this polar vector may be called the polar direction. There is a polar direction corresponding to each point of the curve $a =$ constant.

It should be noted that by this definition of polar vector, the vectors $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are the polar vectors of the parameter curves $v =$ constant and $u =$ constant, respectively.

If this same operation is performed on $\boldsymbol{t}$, $(\boldsymbol{t}a)/[(fa)^2]^{1/2}$ results, which is really $d\boldsymbol{t}/ds$ since $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are $\partial\boldsymbol{t}/\partial u$ and $\partial\boldsymbol{t}/\partial v$, respectively. This vector is orthogonal to the polar vector, because $\boldsymbol{t}^2 = 1$ and $\boldsymbol{t}(d\boldsymbol{t}/ds) = 0$.

The vector $(\boldsymbol{t}a)/[(fa)^2]^{1/2}$ will be called a principal normal to $a =$ constant, relative to the basis, and its magnitude will be known as the first curvature relative to the basis. Usually these expressions will be called simply the principal normal, and the first curvature, the relation to the basis, being understood. In general, as has already been mentioned, a normal vector means a vector orthogonal to the base plane.

Instead of the polar vector $(\boldsymbol{f}a)$ to the curves $a =$ constant, we consider any vector function of u and v in the base plane, say $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2$, where α_1 and α_2 are functions of the coördinates. Suppose there exists a function b such that $\partial b/\partial u = -\lambda\alpha_2$ and $\partial b/\partial v = \lambda\alpha_1$, where λ is a proportionality factor. In this case, $(Vb)/[(fb)^2]^{1/2} = dV/ds$, where s is the arc length of the curve $b =$ constant. $(\boldsymbol{f}b) = \lambda(\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2)$ which is the polar vector to $b =$ constant, and $(Vb) = \lambda(V_1\alpha_1 + V_2\alpha_2)$, hence,

$$(Vb)/[(fb)^2]^{1/2} = (V_1\alpha_1 + V_2\alpha_2)/[(f_1\alpha_1 + f_2\alpha_2)^2]^{1/2}.$$

If $\alpha_1\boldsymbol{f}_1 + \alpha_2\boldsymbol{f}_2$ is the polar vector of a curve $b =$ constant, it follows that

* L. Ingold, *loc. cit.*, p. 583.

the derivatives of any point function with respect to arc length along $b=$ constant is obtained by writing V_1, V_2 (that is $\partial V/\partial u$, $\partial V/\partial v$) instead of f_1, f_2 in the expression for the polar vector and dividing by the length of the vector.

As an example $(f\phi)(\phi a)$ is the polar vector of the orthogonal trajectories of $a=$ constant. If we denote arc length along one of these curves by σ,

$$dV/d\sigma=(V\phi)(\phi a)/[(fa)^2]^{1/2}.$$

We have used the relation $[(f\phi)(\phi a)]^2=(fa)^2$ which is easily shown to be true.

Let
$$k=(f\phi)(\phi a)(f\psi)(\psi a),$$

and let f in the third factor be interchanged with each element of the second factor; then

$$\begin{aligned} k &= (f\phi)(\phi f)(a\psi)(\psi a)+(f\phi)(fa)(\phi\psi)(\psi a) \\ &= 2(\psi a)^2-k. \end{aligned}$$

Thus,
$$k=[(f\phi)(\phi a)^2=(fa)^2.$$

9. *The principal curvature vector.* The polar vector of the curves $a=$ constant may be written either in the form (fa) or $(f\phi)(\psi\phi)(\psi a)$, and from each of these forms it is possible to determine the curvature vector of the curves $a=$ constant.

If we find the curvature vector dt/ds from the equation

$$t=(f\phi)(\psi\phi)(\psi a)/[(fa)^2]^{1/2},$$
$$dt/ds=(\psi\phi)(\psi a)((f\phi)a)/(fa)^2+\text{a vector in the base plane.}$$

The first term is normal to the base plane, and the others lie in the base plane.

The base component is more easily obtained from $t=(fa)/[(fa)^2]^{1/2}$, and in this case

$$\frac{dt}{ds}=\frac{((fa)a)}{(\phi a)^2}-\frac{(\phi a)((\phi a)a)}{[(\phi a)^2]^2}(fa).$$

Let $((fa)a)$ be resolved into its base and normal components,

$$((fa)a)=p(fa)+q(f\phi)(\phi a)+N.$$

Since (fa) and $(f\phi)(\phi a)$ are orthogonal to each other, we may multiply this equation first by (fa) and then by $(f\phi)(\phi a)$ and find that

$$p=\frac{((\psi a)a)(\psi a)}{(fa)^2},\quad q=\frac{((\psi a)a)(\psi\phi)(\phi a)}{(fa)^2}.$$

Now,

$$\frac{dt}{ds} = \frac{((\psi a)a)(\psi\phi)(\phi a)}{[(fa)^2]^2}(f\theta)(\theta a) + N.$$

From these two expressions for the curvature vector the base component and the normal component may be obtained, and

$$\frac{d\mathbf{t}}{ds} = \frac{(\psi\phi)(\phi a)((\psi a)a)}{[(fa)^2]^2}(f\theta)(\theta a) + \frac{(\psi\phi)(\psi a)((f\phi)a)}{(fa)^2}. \tag{20}$$

If we denote the vector of unit length in the direction of $(f\phi)(\phi a)$ by $\boldsymbol{\tau}$ and the normal component of the curvature by $\boldsymbol{\alpha}$, (20) becomes $\Gamma_1 a\boldsymbol{\tau} + \boldsymbol{\alpha}$ where $\Gamma_1 a$ is the differential parameter

$$\frac{(\psi\phi)(\phi a)((\psi a)a)}{[\sqrt{(fa)^2}]^3}.$$

When $f_1 = \partial f/\partial u$, $f_2 = \partial f/\partial v$, the differential parameter $\Gamma_1 a$ is the geodesic curvature of the curves $a =$ constant. Generalizing this idea we call $\Gamma_1 a$ the geodesic curvature of the curves $a =$ constant *relative to the basis* $[f_1, f_2]$.

The curves $a =$ constant determined by $\Gamma_1 a = 0$ will be called the geodesic of the surface, relative to the basis $[f_1, f_2]$. These will be called merely *geodesics* and *geodesic curvature* hereafter, the relation to the basis being understood.

10. *Other curvature vectors.* If we use σ to denote arc length along the orthogonal trajectories of the curves $a =$ constant, we are able to find, by methods similar to those used above, an expression for $d\mathbf{t}/d\sigma$; also, expressions for $d\boldsymbol{\tau}/ds$ and $d\boldsymbol{\tau}/d\sigma$ may be found.

If we let

$$\begin{aligned} \Gamma_2 a &= \frac{(f\phi)(fa)(\theta a)((\phi a)\theta)}{[\Delta_1 a]^{3/2}}, \qquad \boldsymbol{\beta} = \frac{(\phi a)(\theta a)((f\phi)\theta)}{\Delta_1 a}, \\ \boldsymbol{\gamma} &= \frac{(\psi\phi)(\psi a)(\theta a)((f\phi)\theta)}{\Delta_1 a}, \qquad \boldsymbol{\delta} = \frac{(\phi a)((f\phi)a)}{\Delta_1 a}, \end{aligned} \tag{21}$$

where $\Delta_1 a = (fa)^2$, the entire list of curvature vectors may be written

$$\begin{aligned} d\mathbf{t}/ds &= \Gamma_1 a\boldsymbol{\tau} + \boldsymbol{\alpha}, \qquad & d\mathbf{t}/d\sigma &= -\Gamma_2 a\boldsymbol{\tau} + \boldsymbol{\gamma}, \\ d\boldsymbol{\tau}/d\sigma &= \Gamma_2 a\mathbf{t} + \boldsymbol{\beta}, \qquad & d\boldsymbol{\tau}/ds &= -\Gamma_1 a\mathbf{t} + \boldsymbol{\delta}. \end{aligned}$$

In these equations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote the normal components of the curvature vectors of the curve $a =$ constant and its orthogonal trajectory, $\boldsymbol{\gamma}$ denotes the normal component of $d\mathbf{t}/d\sigma$, and $\boldsymbol{\delta}$ the normal component of $d\boldsymbol{\tau}/ds$.

It is obvious that the normal vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are invariant under a change of parameters but depend on the function a.

We can readily obtain relations connecting these vectors.

If in $((\boldsymbol{f}\phi)\phi)(\psi a)$, the last element in the first factor be exchanged with each of the elements of the second factor,

$$((\boldsymbol{f}\phi)\phi)(\psi a) = ((\boldsymbol{f}\phi)\psi)(\phi a) + ((\boldsymbol{f}\phi)a)(\psi\phi).$$

It is recognized immediately that

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = ((\boldsymbol{f}\phi)\phi)(\psi a)^2/\Delta_1 a = ((\boldsymbol{f}\phi)\phi) = \boldsymbol{h}^*. \tag{23}$$

Thus, $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is an invariant proper and is independent of the function a. This is a well-known relation for the ordinary case.

Another relation for the ordinary case is

$$\boldsymbol{\delta} - \boldsymbol{\gamma} = 0.^*$$

In the general representation, this equation no longer holds as we shall now see.

If the numerator of the normal component of $d\boldsymbol{t}/d\sigma$ be simplified,

$$\begin{aligned}(\psi\phi)(\psi a)(\theta a)((\boldsymbol{f}\phi)\theta) &= (\psi\phi)(\theta a)[((\boldsymbol{f}\phi)\psi)(\theta a) + ((\boldsymbol{f}\phi)a)(\psi\phi)] \\ &= (\psi\phi)((\boldsymbol{f}\phi)\psi)(\theta a)^2 + (\psi\phi)(\psi\theta)(\theta a)((\boldsymbol{f}\phi)a)\end{aligned}$$

where the θ of the last factor has been interchanged with each element of the second factor.

In accordance with the notation used in (16),

$$(\psi\phi)(\psi a)\theta a)((\boldsymbol{f}\phi)\theta) = -\boldsymbol{j}(\theta a)^2 + (\phi a)((\boldsymbol{f}\phi)a)$$

and $$(\psi\phi)(\psi a)(\theta a)((\boldsymbol{f}\phi)\theta)/\Delta_1 a = -\boldsymbol{j} + [(\phi a)((\boldsymbol{f}\phi)a)]/\Delta_1 a.$$

Therefore, it is noted in this general case that

$$\boldsymbol{\delta} - \boldsymbol{\gamma} = \boldsymbol{j}. \tag{24}$$

Thus the vector $\boldsymbol{\delta} - \boldsymbol{\gamma}$ is also independent of the function a.

11. *Polar vectors to other curves.* Any vector function which is a polar vector to a curve $a =$ constant on the surface may be written in the form $p\boldsymbol{t} + q\boldsymbol{\tau}$, where p and q are functions of the coördinates. Let $p^2 + q^2 = 1$, then the polar vector at each point of the curve is a unit vector.

In this discussion, we shall let $p\boldsymbol{t} + q\boldsymbol{\tau}$ be the polar vector of the curves

* L. Ingold, *loc. cit.*, p. 588.

$b=$ constant, and shall denote by s and σ the arc length along b and along the orthogonal trajectories of $b=$ constant, respectively.

The curvature vector of the curves defined by this polar vector,

$$p\mathbf{t}+q\boldsymbol{\tau}\equiv p\,\frac{(fa)}{[(fa)^2]^{1/2}}+q\,\frac{(f\phi)(\phi a)}{[(fa)^2]^{1/2}},$$

has the form

$$\begin{aligned}\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{ds}&=p\,\frac{(p\mathbf{t}+q\boldsymbol{\tau},\,a)}{[(fa)^2]^{1/2}}+q\,\frac{(p\mathbf{t}+q\boldsymbol{\tau},\phi)(\phi a)}{[(fa)^2]^{1/2}}\\&=p\,\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{ds_a}+q\,\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{d\sigma_a}\end{aligned}$$

where s_a and σ_a denote arc length along $a=$ constant and the orthogonal trajectories of $a=$ constant, respectively.

We may list the curvature vectors of the curves defined by the orthogonal unit vector, $q\mathbf{t}-p\boldsymbol{\tau}$, and the cross curvatures as follows:

$$\begin{aligned}\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{d\sigma}&=q\,\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{ds_a}-p\,\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{d\sigma_a},\\\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{ds}&=p\,\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{ds_a}+q\,\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{d\sigma_a},\\\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{d\sigma}&=q\,\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{ds_a}-p\,\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{d\sigma_a}.\end{aligned}$$

When these equations are expanded, the curvature vectors become

$$\begin{aligned}\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{ds}&=G_1(q\mathbf{t}-p\boldsymbol{\tau})+p^2\boldsymbol{\alpha}+pq(\boldsymbol{\gamma}+\boldsymbol{\delta})+q^2\boldsymbol{\beta},\\\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{d\sigma}&=G_2(p\mathbf{t}+q\boldsymbol{\tau})+q^2\boldsymbol{\alpha}-(\boldsymbol{\gamma}+\boldsymbol{\delta})+p^2\boldsymbol{\beta},\\\frac{d[q\mathbf{t}-p\boldsymbol{\tau}]}{ds}&=-G_1(p\mathbf{t}+q\boldsymbol{\tau})+pq\boldsymbol{\alpha}+q^2\boldsymbol{\gamma}-p^2\boldsymbol{\delta}-pq\boldsymbol{\beta},\\\frac{d[p\mathbf{t}+q\boldsymbol{\tau}]}{d\sigma}&=-G_2(q\mathbf{t}-p\boldsymbol{\tau})+pq\boldsymbol{\alpha}+q^2\boldsymbol{\delta}-p^2\boldsymbol{\gamma}-pq\boldsymbol{\beta},\end{aligned}\tag{25}$$

where
$$G_1=-\frac{(qa)}{[\Delta_1 a]^{1/2}}+\frac{(p\phi)(\phi a)}{[\Delta_1 a]^{1/2}}-p\Gamma_1+q\Gamma_2,$$

and
$$G_2=-\frac{(pa)}{[\Delta_1 a]^{1/2}}-\frac{(q\phi)(\phi a)}{[\Delta_1 a]^{1/2}}+q\Gamma_1+p\Gamma_2.$$

If $p=\cos\theta$ and $q=\sin\theta$, the normal components of (25) in the new system are:

$$
\begin{aligned}
&\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \cos^2 \theta + (\boldsymbol{\gamma} + \boldsymbol{\delta}) \sin \theta \cos \theta + \boldsymbol{\beta} \sin^2 \theta,\\
(26)\qquad &\bar{\boldsymbol{\beta}} = \boldsymbol{\alpha} \sin^2 \theta - (\boldsymbol{\gamma} + \boldsymbol{\delta}) \sin \theta \cos \theta + \boldsymbol{\beta} \cos^2 \theta,\\
&\bar{\boldsymbol{\gamma}} = (\boldsymbol{\alpha} - \boldsymbol{\beta}) \sin \theta \cos \theta + \boldsymbol{\gamma} \cos^2 \theta - \boldsymbol{\delta} \sin^2 \theta,\\
&\bar{\boldsymbol{\delta}} = (\boldsymbol{\alpha} - \boldsymbol{\beta}) \sin \theta \cos \theta + \boldsymbol{\delta} \cos^2 \theta - \boldsymbol{\gamma} \sin^2 \theta.
\end{aligned}
$$

From this we get

$$\bar{\boldsymbol{\alpha}} + \bar{\boldsymbol{\beta}} = \boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{h}^*,$$

$\boldsymbol{\alpha} + \boldsymbol{\beta}$ is therefore independent of the angle θ.

The vector $\boldsymbol{\delta} - \boldsymbol{\gamma}$ is also invariant, that is

$$\bar{\boldsymbol{\delta}} - \bar{\boldsymbol{\gamma}} = \boldsymbol{\delta} - \boldsymbol{\gamma} = \boldsymbol{j}.$$

In (25) we denoted by G_1 the geodesic curvature of the curves having $p\boldsymbol{t} + q\boldsymbol{\tau}$ as the polar vector and by G_2 the geodesic curvature of the curves having $q\boldsymbol{t} - p\boldsymbol{\tau}$ as the polar vector, i. e.,

$$
\begin{aligned}
(27)\qquad &G_1 = -(dq/ds_a) + (dp/d\sigma_a) - p\Gamma_1 + q\Gamma_2,\\
&G_2 = -(dp/ds_a) - (dq/d\sigma_a) + q\Gamma_1 + p\Gamma_2.
\end{aligned}
$$

If $p = \cos \theta$ and $q = \sin \theta$,

$$
\begin{aligned}
(dp/ds_a) &= -\sin \theta (d\theta/ds_a) = -q(d\theta/ds_a),\\
(dq/d\sigma_a) &= \cos \theta\, (d\theta/d\sigma_a) = p(d\theta/d\sigma_a),\\
(dq/ds_a) &= \cos \theta (d\theta/ds_a) = p(d\theta/ds_a),\\
(dp/d\sigma_a) &= -\sin \theta (d\theta/d\sigma_a) = -q(d\theta/d\sigma_a).
\end{aligned}
$$

We know that

$$(dV/ds) = p(dV/ds_a) + q(dV/d\sigma_a)$$

and

$$(dV/d\sigma) = q(dV/ds_a) - p(dV/d\sigma_a).$$

Therefore, by substituting these results in the equations for G_1 and G_2,

$$
\begin{aligned}
(28)\qquad &G_1 = -(d\theta/ds) - p\Gamma_1 + q\Gamma_2,\\
&G_2 = (d\theta/d\sigma) + q\Gamma_1 + p\Gamma_2.
\end{aligned}
$$

This is a generalization of a known result.*

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* See Eisenhart, *Differential Geometry*, p. 150, Ex. 17.

On the Approximate Solution of Fredholm's Homogeneous Integral Equation.*

By Rachel Blodgett Adams.

Introduction. A number of papers † have appeared since Fredholm's memoir on integral equations which are concerned with the effect on the solution of a change in the kernel. In fact Fredholm ‡ himself obtained a formula for the first variation of the determinant $D(\lambda)$. With the exception of papers by Tricomi § and Viterbi,|| however, the results leave one in doubt as to the bounds of the changes in the solutions corresponding to a given change in the kernel. The former considers only the non-homogeneous Fredholm integral equation, while the latter treats systems of non-homogeneous integral equations of the first and second kinds of the Volterra type. The present paper gives a method for determining bounds ¶ for the change in position of the zeros of the determinant $D(\lambda)$, and the changes in the solutions of the *homogeneous* Fredholm equations

* Presented to the American Mathematical Society, May 1, 1926, under the title, "The solutions of Fredholm's homogeneous integral equation as functionals of the kernel." The results of this paper were obtained in 1921 and are to be found in a manuscript deposited at that time in the library of Radcliffe College. Inequalities were derived there which are practically equivalent to (4), (5), (7), and (8), published later by Tricomi (cf. note § of this page). His inequalities possess certain advantages over those obtained by the writer, first because of his use of an extension of Hadamard's determinent theorem where the writer used that theorem itself, and secondly because of his introduction of the function $\Omega(x)$ and its derivatives.

† H. Block, in "Några variationsformler inom integralekvationernas teori," *Acta Universitatis Lundensis, Lunds Universitets Årsskrift*, Nova Series, Medicin samt Matematiska och Naturvetenskapliga Ämnen, Vol. 7 (1911), pp. 3-34, finds formulas for the successive variations of the resolvent kernel, characteristic values, and fundamental solutions corresponding to a variation in the kernel and incidentally obtains essentially Theorems I and II of this paper. His method, however, gives no bounds for the changes in the solutions.

‡ "Sur une classe d'équations fonctionnelles," *Acta Mathematica*, Vol. 27 (1903), pp. 379-381.

§ "Sulla risoluzione numerica delle equazioni integrali di Fredholm," *Atti della Reale Accademia Nazionale dei Lincei, Rendiconti*, Classe di Scienze Fisiche, Matematiche, e Naturali, Series 5, Vol. 33 (1924), 1° Semestre, pp. 483-486; 2° Semestre, pp. 26-30.

|| "Sulla risoluzione approssimata delle equazioni integrali di Volterra e sulla applicazione di queste allo studio analitico delle curve," *Reale Istituto Lombardo di Scienze e Lettere, Rendiconti*, Series 2, Vol. 45 (1912), pp. 1027-1060.

¶ The inequalities obtained are stronger than they need be theoretically; cf. G. C. Evans' review of W. V. Lovitt's "Linear Integral Equations," *The American Mathematical Monthly*, Vol. 34 (1927), p. 149.

$$(1) \qquad \phi(x) + \lambda \int_a^b K(x,s)\,\phi(s)\,ds = 0$$

$$(2) \qquad \psi(y) + \lambda \int_a^b K(s,y)\,\psi(s)\,ds = 0$$

corresponding to a small change in the kernel, where $K(x,y)$ is real and continuous in the region $a \leq x \leq b$, $a \leq y \leq b$.

It is well known that when $K(x,y)$ is expressible in the form $\sum_{i=1}^{n} \phi_i(x)\,\psi_i(y)$, the solutions of (1) [or (2)], which are in general infinite series involving multiple integrals, are particularly simple.* This fact suggests the possibility of obtaining approximate solutions for (1) [or (2)] by substituting for the given kernel an approximate one of the above type and solving the resulting equation. The results of this paper enable us in most of the interesting cases which arise in practice to determine bounds for the error thus incurred.†

1. *The change in a zero of $D(\lambda)$, $D(\lambda)$ being known.* Let $\lambda = \lambda_0$ be a zero of $D(\lambda)$ of multiplicity n, and let Γ be a circle about λ_0 of radius ρ to be determined later, such that Γ does not contain in its interior or on its boundary any other zero of $D(\lambda)$. Let $\bar{D}(\lambda)$ be the Fredholm determinant corresponding to a new kernel $\bar{K}(x,y)$. Then a necessary and sufficient condition that $D(\lambda)$ and $\bar{D}(\lambda)$ have the same number of zeros within Γ is given by the inequality

$$(3) \qquad (1/2\pi i)\int_\Gamma \left[D'(\lambda)/D(\lambda) - \bar{D}'(\lambda)/\bar{D}(\lambda)\right] d\lambda < 1;$$

it is clear that this means that the integral on the left is zero, but for our purposes it is more convenient to write the condition in the form (3). We shall find an $\epsilon > o$ so small that if

$$|\,K(x,y) - \bar{K}(x,y)\,| < \epsilon \qquad (a \leq x \leq b,\ a \leq y \leq b),$$

the inequality (3) will be satisfied.

Let $|\,K(x,y)\,|$ be $< K$ and $\epsilon \leq E$ in the region $a \leq x \leq b$, $a \leq y \leq b$. Then Tricomi ‡ deduces inequalities which we prefer to write in the following form:

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* T. Lalesco, *Introduction à la Théorie des Équations Intégrales*, Paris (1912), pp. 49-52.

† Cf. Note † p. 144 and Note † p. 148.

‡ *Loc. cit.*, (16). In § 6, Tricomi gives a table of values of the function $\Omega(x)$ correct to four decimal places as x varies from 0 to 1 at intervals of .05.

(4) $$|D(\lambda)-\bar{D}(\lambda)|<\epsilon|\lambda|(b-a)\Omega'[|\lambda|(b-a)(K+E)]$$

(5) $$|D(x,y,\lambda)-\bar{D}(x,y,\lambda)|<\epsilon\{\Omega'[|\lambda|(b-a)(K+E)] + [|\lambda|(b-a)(K+E)]\,\Omega''[|\lambda|(b-a)(K+E)]\}$$

where $\Omega(x)=\sum_{n=0}^{\infty}(n^{n/2}/n!)\,x^n$. Similarly we could prove

(6) $$|D'(\lambda)-\bar{D}'(\lambda)|<\epsilon(b-a)\{|\lambda|(b-a)(K+E) \times\Omega''[|\lambda|(b-a)(K+E)]+\Omega'[|\lambda|(b-a)(K+E)]\}.$$

Tricomi also derives the relations *

(7) $$|D(\lambda)|<\Omega[|\lambda|(b-a)K]$$

(8) $$|D(x,y,\lambda)|<K\Omega'[|\lambda|(b-a)K].$$

To find a lower bound for $D(\lambda)$ on the circumference Γ of radius ρ, we suppose $D(\lambda)$ developed in a power series about the point λ_0; i. e.,

(9) $$D(\lambda)=c_0(\lambda-\lambda_0)^n+c_1(\lambda-\lambda_0)^{n+1}+\cdots = (\lambda-\lambda_0)^n\,[c_0+c_1(\lambda-\lambda_0)+\cdots]$$

where $c_0\neq 0$. Thus if we denote $|c_0|$ by c and a bound for $|c_i|$, $(i=1,2,\ldots)$, by C, and if we choose $\rho<c/(c+2C)$, we have, since $C|\lambda-\lambda_0|/(1-|\lambda-\lambda_0|)$ is $<c/2$,

$$|D(\lambda)/(\lambda-\lambda_0)^n|\geq[c-C\{|\lambda-\lambda_0|+|\lambda-\lambda_0|^2+\cdots\}] = c-C|\lambda-\lambda_0|/(1-|\lambda-\lambda_0|)$$

or

(10) $$|D(\lambda)/(\lambda-\lambda_0)^n|>c/2.$$

Hence on a circle Γ about λ_0 of radius $\rho<c/(c+2C)$, $|D(\lambda)|$ is $>c\rho^n/2$ and furthermore, $D(\lambda)$ will have no second zero in this circle.

From (9) we have also on the circumference of Γ

(11) $$|D(\lambda)|\leq\rho^n[c+C\rho/(1-\rho)].$$

Similarly if we denote $[c+C\rho/(1-\rho)]$ by $F(c,C,\rho)$ we obtain on Γ

(12) $$|D'(\lambda)|\leq n\rho^{n-1}F(c,C,\rho)+\rho^nF'(c,C,\rho),$$

where F' indicates the derivative of F with respect to ρ.

Now $|D'(\lambda)/D(\lambda)-\bar{D}'(\lambda)/\bar{D}(\lambda)|$ may be written in the form

$$\left|\frac{D'(\lambda)[\bar{D}(\lambda)-D(\lambda)]+D(\lambda)[D'(\lambda)-\bar{D}'(\lambda)]}{D(\lambda)\bar{D}(\lambda)}\right|.$$

* *Loc. cit.* These are essentially inequalities (14).

Therefore if we set $L = (|\lambda_0| + \rho)(b - a)(K + E)$ and if ϵ is chosen so as to satisfy the condition

$$(13) \qquad \epsilon < c\rho^n/\{2(|\lambda_0| + \rho)(b - a)\,\Omega'(L)\}$$

we obtain from (4), (6), (10), (11), and (12) the following inequality valid on the circumference of Γ:

$$|D'(\lambda)/D(\lambda) - \bar{D}'(\lambda)/\bar{D}(\lambda)|$$

$$< \frac{\{nF(c,C,\rho) + \rho F'(c,C,\rho)\}\,\epsilon(|\lambda_0| + \rho)(b-a)\,\Omega'(\mathrm{L}) + \rho F(c,C,\rho)\,\epsilon(b-a)\{\mathrm{L}\Omega''(\mathrm{L}) + \Omega'(\mathrm{L})\}}{(c\rho/2)\,\{(c\rho^n/2) - \epsilon(|\lambda_0| + \rho)(b-a)\,\Omega'(\mathrm{L})\}}.$$

It follows that condition (3) will certainly hold if ϵ satisfies the inequality,

$$(14) \quad \epsilon < c^2\rho^n/2(b-a)\,[(|\lambda_0| + \rho)\,\Omega'(\mathrm{L})\,\{c + 2nF(c,C,\rho) + 2\rho F'(c,C,\rho)\} + 2\rho F(c,C,\rho)\{\mathrm{L}\Omega''(\mathrm{L}) + \Omega'(\mathrm{L})\}].$$

Thus $D(\lambda)$ and $\bar{D}(\lambda)$ will have the same number of zeros inside the circle about λ_0 of radius $\rho < c/(c + 2C)$ or in other words $\bar{D}(\lambda)$ will have exactly n zeros $\bar{\lambda}_i$ satisfying the condition $|\lambda_0 - \bar{\lambda}_i| < \rho$.

2. *The change in a zero of $D(\lambda)$, $D(\lambda)$ being unknown.* In practice we do not know $D(\lambda)$, and so cannot determine c and C directly. This difficulty we meet by developing a method of trial and error which gives the desired information.

Let $\bar{\lambda}_1$ be a zero of $\bar{D}(\lambda)$ of multiplicity $\bar{n}$. Consider a circle $\bar{\Gamma}$ with center at $\bar{\lambda}_1$ and radius $\bar{\rho} < \bar{c}/(\bar{c} + 2\bar{C})$, where $\bar{c}$ and $\bar{C}$ have the meanings for $\bar{D}(\lambda)$ that c and C have for $D(\lambda)$. We thus obtain the condition under which $\bar{n}$ zeros of $D(\lambda)$ lie in the circle $|\lambda - \lambda_1| = \bar{\rho}$ by replacing λ_0, ρ, c, C, and n by $\bar{\lambda}_1$, $\bar{\rho}$, $\bar{c}$, $\bar{C}$, and $\bar{n}$ respectively in (14).

If this condition is not fulfilled, we take a better approximation to $K(x,y)$ and try again; in case of a second failure we take a still better approximation, and so on. If the multiplicity of $\bar{\lambda}_1$ is always the same as that of the corresponding zero of $D(\lambda)$, $\bar{c}$ will approach a limit not zero as $\bar{K}(x,y)$ approaches $K(x,y)$ and the above mentioned inequality can eventually be satisfied if we carry this process of repeated trial far enough. If, however, we are unable to satisfy the inequality, we conclude that the zero of $D(\lambda)$ corresponds to several zeros of $\bar{D}(\lambda)$ and so improve the approximation until it is clear that the zeros of $\bar{D}(\lambda)$ are gathering into groups. Suppose for convenience $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ to be such a group.

Accordingly in the neighborhood of $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, we have

$$\bar{D}(\lambda) = (\lambda - \bar{\lambda}_1)^{\bar{n}_1} (\lambda - \bar{\lambda}_2)^{\bar{n}_2} (\lambda - \bar{\lambda}_3)^{\bar{n}_3} [\bar{c}_0 + \bar{c}_1 (\lambda - \bar{\lambda}_1) + \cdots]. \tag{15}$$

In order to get a lower bound for $\lambda - \bar{\lambda}_2$ and $\lambda - \bar{\lambda}_3$ on a circle about $\bar{\lambda}_1$ of radius $\bar{\rho} < \bar{c}/(\bar{c} + 2\bar{C})$, we shall show that for a proper choice of ϵ, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ will lie in a circle of radius $\bar{\rho}/2$ about $\bar{\lambda}_1$.

From the considerations of § 1, we know that n zeros of $\bar{D}(\lambda)$ cluster about a zero of $D(\lambda)$ of multiplicity n. If we are right in our guess that $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$ are such a group, then $D(\lambda)$ will be of the form

$$(\lambda - \lambda_0)^{\bar{n}_1 + \bar{n}_2 + \bar{n}_3} [c_0 + c_1 (\lambda - \lambda_0) + \cdots], \tag{16}$$

and as $\bar{D}(\lambda)$ approaches $D(\lambda)$, $\bar{c}_0$ approaches c_0, $\bar{c}_1$ approaches c_1, etc. Therefore $\bar{c}/2(\bar{c} + 2\bar{C})$ does not grow indefinitely small but approaches a limit not zero; namely, $c/2(c + 2C)$. Therefore for a sufficiently close approximation, we have

$$|\bar{\lambda}_1 - \bar{\lambda}_i| < \bar{\rho}/2, \qquad (i = 2, 3). \tag{17}$$

Furthermore $\bar{c}_0 + \bar{c}_1 (\lambda - \bar{\lambda}_1) + \cdots$ will have no zeros in a circle about $\bar{\lambda}_1$ of radius $\bar{\rho} < \bar{c}/(\bar{c} + 2\bar{C})$ and on the circumference of such a circle we shall have

$$\bar{c}_0 + \bar{c}_1 (\lambda - \bar{\lambda}_1) + \cdots > \bar{c}/2. \tag{18}$$

Thus on this circle we find

$$\begin{aligned} |\bar{D}(\lambda)| &> |(\lambda - \bar{\lambda}_1)^{\bar{n}_1} (\lambda - \bar{\lambda}_2)^{\bar{n}_2} (\lambda - \bar{\lambda}_3)^{\bar{n}_3}| \, \bar{c}/2 \\ &> \bar{c}\bar{\rho}^n / 2^{1 + \bar{n}_2 + \bar{n}_3}, \end{aligned} \tag{19}$$

and $\bar{D}(\lambda)$ will have no zeros other than $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ within the circle.

As in (11) and (12), we have on $\bar{\Gamma}$

$$|\bar{D}(\lambda)| \leq \bar{\rho}^n (3/2)^{\bar{n}_2 + \bar{n}_3} F(\bar{c}, \bar{C}, \bar{\rho}) \tag{20}$$

and

$$|\bar{D}'(\lambda)| \leq \bar{\rho}^{n-1} (3/2)^{\bar{n}_2 + \bar{n}_3} [nF(\bar{c}, \bar{C}, \bar{\rho}) + \bar{\rho}F'(\bar{c}, \bar{C}, \bar{\rho})]. \tag{21}$$

It can be easily verified now that we obtain the condition under which $D(\lambda)$ and $\bar{D}(\lambda)$ have the same number of zeros in the circle of radius $\bar{\rho}$ if in (14), in addition to replacing λ_0, ρ, c, and C by $\bar{\lambda}_1$, $\bar{\rho}$, $\bar{c}$, and $\bar{C}$ respectively, we substitute for $\bar{\rho}^n$, $F(\bar{c}, \bar{C}, \bar{\rho})$ and $F'(\bar{c}, \bar{C}, \bar{\rho})$ the values $\bar{\rho}^n/2^{\bar{n}_2 + \bar{n}_3}$, $3^{\bar{n}_2 + \bar{n}_3} F(\bar{c}, \bar{C}, \bar{\rho})$ and $3^{\bar{n}_2 + \bar{n}_3} F'(\bar{c}, \bar{C}, \bar{\rho})$ respectively.

Now since $\bar{c}$ approaches a definite limit not zero as $\bar{K}(x, y)$ approaches

$K(x,y)$, the inequality mentioned in the preceding paragraph together with (17) can eventually be satisfied. Since $\bar{\lambda}_2$ and $\bar{\lambda}_3$ lie in a circle of radius $\bar{\rho}/2$ about $\bar{\lambda}_1$, we shall then have

$$(22) \qquad | \lambda_0 - \bar{\lambda}_i | < 3\bar{\rho}/2 \qquad (i = 1, 2, 3).$$

3. *The change in the fundamental solutions for the case that the characteristic values, both for $K(x,y)$ and $\bar{K}(x,y)$, are simple poles of the corresponding resolvent kernels.* It is well known that equations (1) and (2) have non-zero solutions when and only when λ is a zero of $D(\lambda)$. For such values of λ the resolvent kernel always has a pole. These zeros of $D(\lambda)$ are called *characteristic values* and the solutions *fundamental solutions.** We shall now determine the change in these solutions, when $K(x,y)$ is replaced by $\bar{K}(x,y)$. We shall first restrict ourselves to the case that, both for $K(x,y)$ and $\bar{K}(x,y)$, the characteristic values are simple poles of the corresponding resolvent kernels. This case, however, includes three large classes of equations; namely, all those whose kernels are symmetric, skew-symmetric, or symmetrisable.† When the resolvent kernel has a simple pole the *index* or number of linearly independent solutions of (1) or (2) is equal to the multiplicity of the corresponding zero of $D(\lambda)$, and conversely.‡

Now let $\bar{\lambda}_i$ of multiplicities $\bar{n}_i$ $(i = 1, 2, \ldots, k; \sum_{i=1}^{k} \bar{n}_i = n)$ be the zeros of $\bar{D}(\lambda)$ corresponding to a zero λ_0 of $D(\lambda)$ of multiplicity n, and let $\bar{\Gamma}$ be a circle of radius $\bar{\rho}$ about $\bar{\lambda}_1$ chosen as in § 2, so that $D(\lambda)$ does not vanish on its circumference and the zeros of $D(\lambda)$ and $\bar{D}(\lambda)$ correspond within it. Then in $\bar{\Gamma}$ we have

$$(23) \qquad \frac{D(x,y,\lambda)}{D(\lambda)} = \frac{\phi_1(x,y)}{\lambda - \lambda_0} + \phi_0(x,y,\lambda)$$

$$(24) \qquad \frac{\bar{D}(x,y,\lambda)}{\bar{D}(\lambda)} = \frac{\bar{\phi}_{11}(x,y)}{\lambda - \bar{\lambda}_1} + \frac{\bar{\phi}_{12}(x,y)}{\lambda - \bar{\lambda}_2} + \cdots + \frac{\bar{\phi}_{1k}(x,y)}{\lambda - \bar{\lambda}_k} + \bar{\phi}_0(x,y,\lambda),$$

where $\phi_0(x,y,\lambda)$ and $\bar{\phi}_0(x,y,\lambda)$ are analytic in λ within and on $\bar{\Gamma}$. Then $\phi_1(x,y)$ and $\phi_{1i}(x,y)$ can be written in the forms §

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* For definitions of terms see Lalesco, *loc. cit.*, pp. 31, 51.

† Cf. Lalesco, *loc. cit.*, pp. 66, 74, 81. It is interesting to note that the integral equations arising out of the generalized first and second boundary value problems both in the plane and in space can be treated by the method of this section. Cf. J. Plemelj, "Über lineare Randwertaufgaben der Potentialtheorie," I Teil, *Monatshefte für Mathematik und Physik*, Vol. 15 (1904), pp. 374-383.

‡ Lalesco, *loc. cit.*, p. 57.

§ Lalesco, *loc. cit.*, p. 58.

$$(25) \qquad \phi_1(x, y) = \sum_{j=1}^{n} \phi_j(x)\, \psi_j(y)$$

$$(26) \qquad \bar{\phi}_{1i}(x, y) = \sum_{j=1}^{\bar{n}_i} \bar{\phi}_{ij}(x)\, \bar{\psi}_{ij}(y) \qquad (i = 1, 2, \cdots, k),$$

where the ϕ's and $\bar{\phi}$'s are solutions of (1) for $K(x, y)$ and $\bar{K}(x, y)$ respectively and similarly the ψ's and $\bar{\psi}$'s are solutions of the adjoint equation (2). Also we have *

$$(27) \qquad \int_a^b \phi_i(x)\, \psi_j(x)\, dx = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad (i, j = 1, 2, \cdots, n)$$

$$(28) \qquad \int_a^b \bar{\phi}_{ij}(x)\, \bar{\psi}_{pl}(x)\, dx = \begin{cases} 1, & i = p, \quad j = l \\ & i \neq p \quad \text{or} \\ 0, & \\ & i = p, \quad j \neq l \end{cases} \qquad \begin{pmatrix} i, p = 1, 2, \cdots, k \\ j = 1, 2, \cdots, \bar{n}_i \\ l = 1, 2, \cdots, \bar{n}_p \end{pmatrix}.$$

From their method of formation it is clear that the ϕ's and $\bar{\phi}$'s are finite and not identically zero.† Accordingly we may replace $\bar{\phi}_{ij}(x)$ by $\bar{\phi}_{ij}(x) / (\int_a^b |\bar{\phi}_{ij}(x)|^2 dx)^{1/2}$ and $\bar{\psi}_{ij}(y)$ by $\bar{\psi}_{ij}(y) (\int_a^b |\bar{\phi}_{ij}(x)|^2 dx)^{1/2}$. We shall say that the $\bar{\phi}$'s are then *normalized.* Evidently this definition reduces to the usual one for real functions. We observe that the relations (26) and (28) remain unaltered. Similarly we may normalize the ϕ's.

By the theory of residues we have

$$(29) \qquad (1/2\pi i) \int_{\bar{\Gamma}} [D(x, y, \lambda)/D(\lambda)]\, d\lambda = \sum_{j=1}^{n} \phi_j(x)\, \psi_j(y)$$

and

$$(30) \qquad (1/2\pi i) \int_{\bar{\Gamma}} [\bar{D}(x, y, \lambda)/\bar{D}(\lambda)]\, d\lambda = \sum_{i=1}^{k} \sum_{j=1}^{\bar{n}_i} \bar{\phi}_{ij}(x)\, \bar{\psi}_{ij}(y).$$

Since this last sum has just n terms, we shall write it for convenience in the form

$$(31) \qquad \sum_{j=1}^{n} \bar{\phi}_j(x)\, \bar{\psi}_j(y),$$

the terms being ordered as in (30).

We shall show that given an η_1, an ϵ can be found such that, if $|K(x, y) - \bar{K}(x, y)|$ is $< \epsilon$ in the region $a \leq x \leq b$, $a \leq y \leq b$, we shall have

$$(32) \qquad \left| \sum_{j=1}^{n} \phi_j(x)\, \psi_j(y) - \sum_{j=1}^{n} \bar{\phi}_j(x)\, \bar{\psi}_j(y) \right| < \eta_1.$$

We already have on the circumference of $\bar{\Gamma}$, as in (19),

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* Lalesco, *loc. cit.*, pp. 54, 56.

† Lalesco, *loc. cit.*, pp. 38-40, 47, 52 § 9.

(33) $\quad |\bar{D}(\lambda)| > \bar{c}\bar{\rho}^n/2^{1+\bar{n}_2+\cdots+\bar{n}_k}$,

where

$$\bar{D}(\lambda) = (\lambda - \bar{\lambda}_1)^{\bar{n}_1}(\lambda - \bar{\lambda}_2)^{\bar{n}_2}\cdots$$
$$\times(\lambda - \bar{\lambda}_k)^{\bar{n}_k}[\bar{c}_0 + \bar{c}_1(\lambda - \bar{\lambda}_1) + \cdots].$$

If, therefore, we set $\bar{\mathrm{L}} = (|\bar{\lambda}_1| + \bar{\rho})(b-a)(K+E)$ and if ϵ is chosen to satisfy the condition,

$$\epsilon < \bar{c}\bar{\rho}^n/\{2^{1+\bar{n}_2+\cdots+\bar{n}_k}(|\bar{\lambda}_1| + \bar{\rho})(b-a)\,\Omega'(\bar{\mathrm{L}})\},$$

we obtain from (4), (5), (7), (8), and (33) the inequality,

$$(34)\qquad |(1/2\pi i)\int_{\bar{\Gamma}}[D(x,y,\lambda)/D(\lambda) - \bar{D}(x,y,\lambda)/\bar{D}(\lambda)]\,d\lambda| =$$

$$\left|(1/2\pi i)\int_{\bar{\Gamma}}\frac{D(x,y,\lambda)[\bar{D}(\lambda)-D(\lambda)] + D(\lambda)[D(x,y,\lambda)-\bar{D}(x,y,\lambda)]}{D(\lambda)\,\bar{D}(\lambda)}d\lambda\right|$$

$$< \frac{2^{2(n-\bar{n}_1+1)}\epsilon\bar{\rho}\{\bar{L}[\Omega'(\bar{L})]^2 + \Omega(\bar{L})\Omega'(\bar{L}) + \bar{L}\Omega(\bar{L})\Omega''(\bar{L})\}}{\bar{c}^2\rho^{2n} - 2^{n-\bar{n}_1+1}\bar{c}\bar{\rho}^n\epsilon(|\bar{\lambda}_1| + \bar{\rho})(b-a)\,\Omega'(\bar{L})}.$$

It follows that (32) will hold if ϵ is made to satisfy the inequality,

$$(35)\quad \epsilon < \bar{c}^2\bar{\rho}^{2n-1}\eta_1/\Big[2^{2(n-\bar{n}_1+1)}\{\bar{L}[\Omega'(\bar{L})]^2 + \Omega(L)\Omega'(\bar{L})$$
$$+\bar{L}\Omega(\bar{L})\Omega''(\bar{L})\} + 2^{n-\bar{n}_1+1}\bar{c}\bar{\rho}^{n-1}\eta_1(|\bar{\lambda}_1| + \bar{\rho})(b-a)\Omega'(\bar{L})\Big].$$

But we know that $\bar{c}$ approaches a limit not zero as $\bar{K}(x, y)$ approaches $K(x,y)$. Therefore, as on pp. 143-4, an ϵ can be found to satisfy (35).

Finally we shall show that given an η_2, an η_1 and consequently an ϵ can be found such that each $\bar{\phi}_p(x)$ $(p = 1, 2, \cdots, n)$ differs from a linear combination of the $\phi_p(x)$ by less than η_2, or in other words, each $\bar{\phi}_p(x)$ differs from a solution of (1) by less than η_2. By Schwartz' inequality * we have

$$\int_a^b |\bar{K}(x,s)|\,|\bar{\phi}_p(s)|\,ds \leq |(\int_a^b |\bar{\phi}_p(s)|^2 ds \int_a^b |\bar{K}(x,s)|^2 ds)^{1/2}|$$
$$\leq |(\int_a^b |\bar{K}(x,s)|^2 ds)^{1/2}|$$
$$< |\bar{K}(b-a)^{1/2}|,$$

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* E. Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Mathematische Annalen*, Vol. 63 (1907), p. 440.

where $|\bar{K}(x,y)|$ is $< \bar{K}$ in the region $a \leq x \leq b$, $a \leq y \leq b$. If we denote by $\bar{\lambda}$ a bound for $|\bar{\lambda}_i|$, $(i = 1, 2, \ldots, k)$, we then have

$$|\bar{\phi}_p(x)| \leq \bar{\lambda} \int_a^b |\bar{K}(x,s)|\,|\bar{\phi}_p(s)|\,ds$$
$$< \bar{\lambda}\bar{K}\,|(b-a)^{1/2}| \qquad (p = 1, 2, \cdots, n).$$

Multiplying (32) by $|\bar{\phi}_p(y)|$ and integrating with respect to y, we obtain

$$(36) \quad \left|\sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j(y)\,\bar{\phi}_p(y)\,dy - \bar{\phi}_p(x)\right| < \eta_1 \bar{K}\bar{\lambda}\,|(b-a)^{3/2}|.$$

Consequently the left hand member of (36) is less than η_2 provided we have

$$\eta_1 \bar{K}\bar{\lambda}\,|(b-a)^{3/2}| < \eta_2$$

or

$$(37) \qquad \eta_1 < \eta_2/\{\bar{K}\,\bar{\lambda}\,|(b-a)^{3/2}|\}.$$

We see that if ϵ is made to satisfy (35) and (37) in addition to (17) and the other inequality mentioned in § 2, we have

$$(38) \quad \left|\sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j(y)\,\bar{\phi}_p(y)\,dy - \bar{\phi}_p(x)\right| < \eta_2 \qquad (p = 1, 2, \cdots, n).$$

In the same manner we can show that given an η_3, an ϵ can be found such that each $\phi_p(x)$ differs by less than η_3 from a suitable linear combination of the $\bar{\phi}_p(x)$. A similar discussion is possible for the $\psi_p(y)$ and $\bar{\psi}_p(y)$. Thus we have proved

Theorem I. *Let λ_0 of multiplicity n and $\bar{\lambda}_i$ $(i = 1, 2, \cdots, k)$ of multiplicities $\bar{n}_i$ be corresponding characteristic values for $K(x, y)$ and $\bar{K}(x, y)$ respectively. If the corresponding resolvent kernels have only simple poles at these points, then given an arbitrarily small positive quantity δ, a positive number ϵ can be found such that, if $|K(x, y) - \bar{K}(x, y)|$ is $< \epsilon$ in the region $a \leq x \leq b$, $a \leq y \leq b$, the difference between each of the n normalized fundamental solutions of* (1) [*or* (2)] *for either kernel and a suitable linear combination of the n normalized fundamental solutions for the other is in absolute value less than δ.*

4. *The change in the fundamental solutions for the general case.* We shall consider briefly the general case in which no hypothesis is made regarding the orders of the poles of the resolvent kernels. As before let λ_0 be a zero of $D(\lambda)$ of multiplicity n, and $\bar{\lambda}_i$, of multiplicities $\bar{n}_i$ $(i = 1, 2, \cdots, k)$, be the corresponding zeros of $\bar{D}(\lambda)$.

We have *

$$(1/2\pi i) \int_{\Gamma} [D(x, y, \lambda)/D(\lambda)]\,d\lambda = \sum_{j=1}^{n} \phi_j(x)\,\psi_j(y),$$

* Lalesco, *loc. cit.*, pp. 54-57.

where the ϕ's and ψ's are called *fundamental functions*, r of the ϕ's being solutions of (1) and r of the ψ's solutions of (2) for $\lambda = \lambda_0$. Similarly, we have

$$(1/2\pi i)\int_{\bar{\Gamma}} [\bar{D}(x,y,\lambda)/\bar{D}(\lambda)]\, d\lambda = \sum_{j=1}^{n} \bar{\phi}_j(x)\, \bar{\psi}_j(y),$$

where the $\bar{\phi}$'s and $\bar{\psi}$'s are fundamental functions for $\bar{K}(x,y)$, $\bar{r}$ of the $\bar{\phi}$'s and $\bar{\psi}$'s being solutions of (1) and (2) respectively for $\bar{K}(x,y)$. As in § 3, the ϕ's and ψ's form a biorthogonal system *; so also do the $\bar{\phi}$'s and $\bar{\psi}$'s. It is a simple matter to normalize the ϕ's and $\bar{\phi}$'s; in what follows we shall suppose this to have been done.

Exactly as before we can show that given an η, an ϵ can be found such that, for

$$| K(x,y) - \bar{K}(x,y) | < \epsilon \qquad (a \leq x \leq b,\ a \leq y \leq b),$$

we shall have

$$| \sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j(y)\, \bar{\phi}_p(y)\, dy - \bar{\phi}_p(x) | < \eta, \tag{39}$$

where $\bar{\phi}_p(x)$ is any one of the $\bar{r}$ fundamental solutions of (1) for $\bar{K}(x,y)$. Evidently this statement still holds if we interchange ϕ and $\bar{\phi}$, ψ and $\bar{\psi}$ and replace $\bar{r}$ and $\bar{K}(x,y)$ by r and $K(x,y)$ respectively. A similar discussion is possible for the $\psi_p(y)$ and $\bar{\psi}_p(y)$. Thus a proof is obtained for †

THEOREM II. *Let λ_0 of multiplicity n and $\bar{\lambda}_i$ $(i = 1, 2, \cdots, k)$ of multiplicities $\bar{n}_i$ be corresponding characteristic values for $K(x,y)$ and $\bar{K}(x,y)$ respectively. If no hypothesis is made regarding the orders of the poles of the resolvent kernels, then given an arbitrarily small positive quantity δ, a positive number ϵ can be found such that, if $| K(x,y) - \bar{K}(x,y) |$ is $< \epsilon$ in the region $a \leq x \leq b$, $a \leq y \leq b$, the difference between each of the normalized fundamental solutions of* (1) [*or* (2)] *for either kernel and a suitable linear combination of the n normalized* fundamental functions *for the other is in absolute value less than δ.*

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* Lalesco, *loc. cit.*, pp. 48, 54, 56.

† Since not every linear combination of the ϕ's is a solution of (1), we do not know here as in § 3 that each $\bar{\phi}_p(x)$ differs from a solution of (1) by less than η. Furthermore the inequality obtained from (39) by interchanging ϕ and $\bar{\phi}$, ψ and $\bar{\psi}$ does not enable us actually to find approximate solutions of (1) since we cannot calculate $\int_a^b \bar{\psi}_j(y)\phi_p(y)\,dy$. We simply know such approximate solutions exist.

Functions of *Écart Fini.*

BY H. E. BRAY.

1. *Introduction.* A function $f(x)$ is said to be of *écart fini* * in the interval $-\pi \leq x \leq \pi$, if it is summable (Lebesgue) and if the integrals

$$\int_a^b f(x) n \sin nx \, dx, \quad \int_a^b f(x) n \cos nx \, dx$$

are bounded, independently of the integer n and the numbers a, b; where $-\pi \leq a < b \leq \pi$. This property is of importance in the study of the behavior of an analytic function in the neighborhood of its circle of convergence.†

In this paper we shall study some of the interesting conditions which are necessary or sufficient in order that a given function be of écart fini.

Without loss of generality we shall assume that $f(x)$ is periodic, of period 2π; then the property stated above is equivalent to the statement that the quantity,

$$\int_a^b f(x+k) \cos nx \, dx$$

is uniformly of the order $(1/n$, i. e., $O(1/n)$, for all n, k, a, b, where $|b-a| \leq 2\pi$. In fact n need not be an integer; for if $n = N + \theta$, where N is an integer and $0 \leq \theta \leq 1$ we have after integrating by parts

$$\begin{aligned} \int_a^b f(x) n \sin nx \, dx &= \int_a^b f(x)(N+\theta)(\sin Nx \cos \theta x + \cos Nx \sin \theta x) dx \\ &= (N+\theta) \left[\cos \theta x \int_x^{-\pi} f(x) \sin Nx \, dx \right]_a^b \\ &\quad + (N+\theta) \int^b \left\{ \int_{-\pi}^x f(t) \sin Nt \, dt \right\} \theta \sin \theta x \, d\theta \\ &\quad + \cdots \end{aligned}$$

Consequently if our integral is bounded when $n = N$, it is also bounded when $n = N + \theta$.

2. *Elementary properties.* We first recall the following elementary property:

THEOREM 1. *Every function $f(x)$, of bounded variation, $-\pi \leq x \leq \pi$, is a function of écart fini.*

* Hadamard, *Journal de Liouville*, Ser. 4, Vol. 8 (1892), p. 154 et seq.

† Mandelbrojt, *The Rice Institute Pamphlet*, Vol. 14 (Oct., 1927), No. 4, p. 293 et seq.

For on integrating by parts, we have

$$\int_a^b f(x+k)n\cos nx\,dx = \sin nx\, f(x+k)\Big]_a^b + \int_a^b \sin nx\, d_x f(x+k).$$

The Stieltjes integral in the second member is numerically $\leqq T$, the total variation of f in the interval $-\pi \leqq x \leqq \pi$. Since the other term is evidently bounded, the theorem is proved.

It was this theorem which suggested to Hadamard the converse question, whether a function of écart fini is necessarily of bounded variation. We shall show later that this question must be answered in the negative, even if we consider continuous functions exclusively. As a corollary of the foregoing theorem we have the result:

If $f(x, \alpha)$ is uniformly of bounded variation in x, and uniformly bounded for all values of the parameter α, then $f(x, \alpha)$ is uniformly of écart fini.

THEOREM 2. *If $f(x)$ is of écart fini it is equivalent to a bounded function.*

We shall show, in fact, that the derivative of $F(x)$, the integral of $f(x)$, is bounded wherever this derivative exists, i. e. almost everywhere; from which it will follow that, except on a point set of measure zero, $f(x)$ is bounded.

Consider then the integral

$$\int_0^{\pi/2n} f(x+t)n\cos nt\,dt$$

which, by hypothesis, is bounded for all values of n and x. On integrating by parts we obtain, after setting $F(x+t) = F(x) + \int_0^t f(x+t)dt$,

$$-nF(x) + \int_0^{\pi/2n} F(x+t)n^2\sin nt\,dt$$
$$= \int_0^{\pi/2n} [F(x+t) - F(x)]\, n^2 \sin nt\,dt.$$

If now we suppose that the derivative of $F(x)$ exists at the point x and is equal to $f(x)$ we may replace $F(x+t) - F(x)$ by $t(f(x) + \eta(t))$ where $\eta(t)$ approaches zero with t. We obtain on integrating

$$\int_0^{\pi/2n} (f(x) + \eta(t))\, t\, n^2 \sin nt\,dt$$
$$= f(x) + \int_0^{\pi/2n} \eta(t)\; t\, n^2 \sin nt\,dt.$$

The last integral is not greater numerically than the upper bound of $\eta(t)$ in the interval $[0, \pi/2n]$, and it approaches zero as $n \rightarrow \infty$, since

$$\int_0^{\pi/2n} t\, n^2 \sin nt\,dt = 1.$$

Hence $f(x)$ is bounded at all points where $f(x) = dF(x)/dx$.

It is interesting to see whether the last theorem will be valid if we set aside the assumption that $f(x)$ is summable in the Lebesgue sense and assume merely that $f(x)$ is measurable and integrable in the Denjoy sense, and that the *Denjoy integral*

$$\int_a^b f(x+t)\,n\cos nt\,dt$$

is bounded for all n, x, a, b, $a \leq b \leq a + 2\pi$. Since Denjoy integrals are not in general absolutely convergent integrals it is conceivable that this integral could be bounded and yet permit $f(x)$ not to be equivalent to a bounded function. We shall show, however, that this is not possible. In our proof, we require the following lemma:

LEMMA. *If $f(x)$ is integrable (D) and if $\alpha(x)$ is a function of bounded variation, then*

$$\int_c^d dx \int_a^b f(x+t)\alpha(t)\,dt = \int_a^b \alpha(t)\,dt \int_c^d f(x+t)\,dx.$$

Since a Denjoy integral is a continuous function of its limits of integration the integral $\int_c^d f(x+t)\,dx = F(d+t) - F(c+t)$ is a continuous function of t. Let us approximate to $\alpha(t)$ by means of a bounded step function $\alpha_n(t)$ of uniformly bounded variation, where $\alpha_n(t)$ is constant on each of a finite number k_n of intervals, for each n; and $\lim \alpha_n(t) = \alpha(t)$; $a \leq t \leq b$.

Since the inner integral is continuous and $\alpha_n(t)$ is bounded we have, first,

$$\lim_{n=\infty} \int_a^b dt\,\alpha_n(t) \int_c^d f(x+t)\,dx = \int_a^b dt\,\alpha(t) \int_c^d f(x+t)\,dx.$$

Let

$$\begin{aligned} \alpha_n(t) &= a_i^{(n)} \text{ when } t_i^{(n)} \leq t < t_{i+1}^{(n)},\ i = 1, 2, \cdots k_{(n-1)} \\ &= a_{k_n}^{(n)} \text{ when } t_{k_n}^{(n)} \leq t \leq t_{k_{n+1}}^{(n)}, \\ &\text{where } t_1^{(n)} = a,\ t_{k_n+1}^{(n)} = b,\quad n = 1, 2, 3, \cdots. \end{aligned}$$

Then

$$\begin{aligned} \int_a^b dt\,\alpha_n(t) \int_c^d f(x+t)\,dx &= \sum_{i=1}^{k_n} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} dt\,a_i^{(n)} \int_c^d f(x+t)\,dx \\ &= \int_c^d dx \Big(\sum_{i=1}^{k_n} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} a_i^{(n)} f(x+t)\,dx\Big) \\ &= \int_c^d dx \int_a^b \alpha_n(t) f(x+t)\,dt \end{aligned}$$

$$= \int_c^d dx \left\{ [\alpha_n(b)F(x+b) - \alpha_n(a)F(x+a)] - \int_a^b F(x+t)\,d\alpha_n(t) \right\}$$

the last step being obtained by the valid process of integrating by parts.* Now, as n becomes infinite, each term within the square bracket approaches boundedly a continuous function of x, and the Stieltjes integral approaches boundedly the Stieltjes integral in which $\alpha_n(t)$ is replaced by $\alpha(t)$, on account of the hypothesis on α_n.† Consequently

$$\lim \int_a^b dt\, \alpha_n(t) \int_c^d f(x+t)\,dx$$

$$= \int_d^c dx \left\{ \alpha(b)F(x+b) - \alpha(a)F(x+a) - \int_a^b F(x+t)\,d\alpha(t) \right\}$$

$$= \int_c^d dx \int_a^b \alpha(t) f(x+t)\,dt.$$

Hence

$$\int_a^b dt\, \alpha(t) \int_c^d f(x+t)\,dx = \int_c^d dx \int_a^b f(x+t)\alpha(t)\,dt.$$

In order to prove that $f(x)$ is nearly everywhere bounded assuming that it is of écart fini in the wider (Denjoy) sense we proceed as follows. Consider the approximating function

$$f_\mu(x) = (1/\mu) \int_0^\mu f(x+t)\,dt.$$

It is well known that $\lim\limits_{\mu=0} f_\mu(x) = f(x)$ almost everywhere when $f(x)$ is integrable (D),‡ and also that $f_\mu(x)$ is a continuous function of x for each $\mu > 0$. Also

$$\int_a^b f_\mu(x+t)\, n \cos nt\, dt$$

$$= (1/\mu) \int_a^b n \cos nt\, dt \int_0^\mu f(x+t+t')\,dt'$$

$$= (1/\mu) \int_0^\mu dt' \int_a^b f(x+t+t')\, n \cos nt\, dt$$

by the foregoing lemma. But the inner integral is, by hypothesis, uniformly bounded for every value of t', x, a, b. Hence its average value in the interval $0 \leq t' \leq \mu$ is uniformly bounded, which signifies that the function $f_\mu(x)$ is uniformly of écart fini. It follows, therefore, by Theorem 2, that $f_\mu(x)$ is

* Hobson, *The Theory of Functions of a Real Variable*, Second Edition, Vol. 1, p. 648.

† H. E. Bray, *Annals of Mathematics*, Vol. 20 (1919), p. 180.

‡ Hobson, *loc. cit.*, p. 629 et seq.

uniformly bounded for all μ. Hence $f(x)$ which is equal almost everywhere to $\lim_{\mu=0} f_\mu(x)$ is bounded except on a set of measure zero. We have now proved that every function of écart fini in the Denjoy sense is nearly everywhere bounded, therefore integrable (L), and of écart fini in the ordinary sense.

3. *Necessary and sufficient condition.* In the next theorem we shall show that the condition that $f(x)$ be a function of écart fini may be expressed in terms of $F(x)$, the indefinite integral of $f(x)$, without introducing explicitly the trigonometric functions used in the definition.

Let δ be any positive number $\leq 2\pi$ and let m be any positive integer such that $m\delta \leq 2\pi$, and let

$$E(x, m, \delta) = \sum_{r=1}^{m} (-1)^r \{F(x + r\delta) - F(x + \overline{r-1}\delta)\}$$
$$= F(x) - 2F(x + \delta) + 2F(x + 2\delta) - \ldots + (-1)^m F(x + md).$$

THEOREM 3. *A necessary and sufficient condition in order that $f(x)$ be of écart fini is that $E(x, m, \delta)$ be uniformly of the order δ for all pairs of values of m, δ, subject to the condition $m\delta \leq 2\pi$ and for every x, $-\pi \leq x \leq \pi$.*

To prove that this condition is sufficient we observe first that if $F(x + \delta) - F(x) = O(\delta)$ it follows that $f(x)$ is bounded nearly everywhere and therefore in considering the boundedness of the quantity $n \int_a^b f(x + t) \cos nt\, dt$ a part of the interval $[a, b]$ whose length is $O(1/n)$ may be disregarded. We therefore consider the integral

$$\int_0^a f(x + t) \cos nt\, dt,$$

for our integral is the difference of two integrals of this kind, and let $a = m\pi/n$. We have, on integrating by parts,

$$\int_0^{m\pi/n} f(x + t) \cos nt\, dt = (-1)^m F(x + m\pi/n) - F(x)$$
$$+ \int_0^{m\pi/n} F(x + t) n \sin nt\, dt.$$

But

$$\int_0^{m\pi/n} F(x + t) n \sin nt\, dt$$
$$= (1/2) \int_0^{\pi/n} [F(x + t) + (-1)^{m-1} F(x + t + (m-1)\pi/n)] n \sin nt\, dt$$
$$+ (1/2) \int_0^{\pi/n} E(x + t, m, \pi/n) n \sin nt\, dt.$$

Since $E(x+t, m, \pi/n)$ is $O(1/n)$ the last integral is $O(1/n)$ and we can write

$$\int_0^{m\pi/n} f(x+t) \cos nt\, dt$$

$$= (1/2) \int_0^{\pi/n} [F(x+t) + (-1)^{m-1} F(x+t+(m-1)\pi/n)] n \sin nt\, dt$$
$$- F(x) + (-1)^m F(x + m\pi/n) + O(1/n)$$
$$= (1/2) \int_0^{\pi/n} [F(x+t) - F(x)] n \sin nt\, dt$$
$$+ \frac{(-1)^{m-1}}{2} \int_0^{\pi/n} [F(x+t+(m-1)\pi/n) - F(x+m\pi/n)] n \sin nt\, dt$$
$$+ O(1/n).$$

In the last two integrals the integrands are both bounded, since in the first integral, for example, $F(x+t) - F(x) = O(t) = O(1/n)$, since $0 \leq t \leq \pi/n$. Hence the last expression is $O(1/n)$. This establishes the sufficiency of the condition.

To show that our condition is necessary, we consider the periodic function $p(x)$ of period 2π, defined in the interval $-\pi \leq x \leq \pi$ as follows:

$$\begin{aligned} p(x) &= 1, & 0 < x < \pi \\ &= -1, & -\pi < x < 0 \\ &= 0 & \text{otherwise.} \end{aligned}$$

The Fourier expansion for $p(x)$ is given by

$$p(x) = (4/\pi) \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin (2m-1)x$$

and if n is a positive number, not necessarily an integer,

$$p(n\pi x) = (4/\pi) \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin n(2m-1)\pi x$$

and since $p(x)$ is of bounded variation we can calculate by termwise integration the integral

$$\int_a^b f(k+x) p(n\pi x)\, dx = (4/\pi) \sum_{m=1}^{\infty} \frac{1}{2m-1} \int_a^b \sin n(2m-1)\pi x\, f(k+x)\, dx.$$

Therefore

$$\left| \int_a^a f(k+x) p(n\pi x)\, dx \right| \leqq (4/n\pi) \sum_{m=1}^{\infty} \frac{K}{(2m-1)^2} \leq \frac{8K}{n\pi}$$

since the respective integrals in the second member are uniformly of the order $1/(n(2m-1))$. Consequently

$$E(k, r, 1/n) = \int_0^{r/n} f(k+x)\,p(n\pi x)\,dx$$

is of the order $1/n$, i. e. $E(k, r, \delta)$ is $O(\delta)$. The theorem is now proved.

The method used in proving Theorem 3 is applicable, if $f(x)$ is assumed to be bounded, to the case where the index m is such that the length of the interval of integration $m\pi/n$ is equal to 2π. We thus obtain a necessary and sufficient condition that the integral

$$\int_0^{2\pi} f(k+x)\,n\cos nx\,dx$$

remain bounded, in other words, a necessary and sufficient condition that the Fourier coefficients of a bounded measurable function be of the order $1/n$. We may therefore state without further proof:

THEOREM 4. *If $f(x)$ is bounded and measurable, then a necessary and sufficient condition that its Fourier coefficients,*

$$a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

be of the order $1/n$, is that the quantity

$$E(x, 2n, \pi/n) = F(x) - 2F(x+\pi/n) + 2F(x+2\pi/n) - \dots \\ - 2F(x+(2n-1)\pi/n) + F(x+2\pi)$$

be of the order $1/n$, uniformly for every x.

4. *Necessary conditions.* It is interesting to observe that in the latter part of the proof of Theorem 3, the only properties required of the function denoted by $p(x)$ are these:

i) $p(x)$ is periodic, of period 2π.

ii) $p(x)$ is of bounded variation in the interval $-\pi \leq x \leq \pi$, its total variation being $\leq T$,

iii) $\int_{-\pi}^{\pi} p(x)\,dx = 0.$

It then follows that the Fourier coefficients of $p(x)$ satisfy the inequalities

$$|a_k| \leq T/\pi k, \qquad |b_k| \leq T/\pi k.$$

As before, we have

$$\int_a^b f(x)\,p(nx)\,dx = \sum_{k=1}^{\infty}\int_a^b f(x)\,(a_k\cos knx + b_k\sin knx)\,dx$$

and, if $f(x)$ is of écart fini, it follows that a constant K exists, such that

$$\left| \int_a^b f(x) \cos nx \, dx \right| \leq K/n, \quad \left| \int_a^b f(x) \sin nx \, dx \right| \leq K/n.$$

Consequently,

$$\left| \int_a^b f(x) p(nx) dx \right| \leq \frac{2TK}{\pi n} \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{4TK}{\pi n}.$$

We thus obtain

THEOREM 5. *A necessary condition in order that $f(x)$ be of écart fini, is that a constant K exist, such that*

$$\left| \int_a^b f(x) p(nx) dx \right| \leq 4TK/n\pi$$

where $p(x)$ is any function satisfying conditions i), ii) and iii).

5. *Sufficient conditions.* In order to show that a function of écart fini is not in general of bounded variation, even if continuous, nor equivalent to a function of bounded variation, we proceed to establish sufficient conditions which are easily applicable to a wide class of functions. These conditions apply to functions $f(x)$ possessing derivatives of the first order which are integrable, except in the neighborhood of a finite number of points in the interval $-\pi \leq x \leq \pi$. For example, we shall show that every function of the form $x^a \sin x^{-a}$, $\alpha > 0$, is of écart fini though its derivative is not summable in the whole interval $-\pi \leq x \leq \pi$ and therefore the function itself is not of bounded variation in the whole interval. In any interval which contains the point $x = 0$, the function is of unbounded variation. For the sake of clearness we will state our theorems for the case in which there is just one point in the neighborhood of which the function $f(x)$ is of unbounded variation. This point we shall take as the point $x = 0$.

THEOREM 6. *If $g(x)$ is a function of finite variation in the interval $\epsilon \leq x \leq 2\pi - \epsilon$; and if the total variation of $g(x)$ in this interval is $O(1/\epsilon)$; and if $f(x)$ is bounded and satisfies the equation,*

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t) dt, \qquad 0 < x_0 \leqq x < 2\pi,$$

then $f(x)$ is of écart fini.

Consider the integral

$$\int_a^b f(x) n \cos n(x + k) dx.$$

Since $f(x)$ is bounded, we may discard a portion of the interval $[a, b]$ of

magnitude of the order $1/n$. Hence we may assume $1/n \leq a < b \leq 2\pi - 1/n$. Integrating by parts we have

$$\int_a^b f(x) n \cos n(x+k)\, dx = f(x) \sin n(x+k) \Big] - \int_a^b \sin n(x+k) g(x)\, dx.$$

The term between limits is bounded since $f(x)$ is bounded. On integrating the second term by parts we obtain

$$\int_a^b \sin n(x+k) g(x)\, dx = - \cos n(x+k) \cdot g(x)/n \Big]_a^b$$
$$+ (1/n) \int_a^b \cos n(x+k)\, dg(x).$$

The term between limits is again bounded, since $g(x)$ is at most of the order n, its total variation being of the order n. The last integral is at most equal to $1/n$ times the total variation of $g(x)$ and is therefore bounded. The theorem is thus proved.

As an example of this type of function, consider

$$f(x) = \sin \log (1/x), \qquad 0 < x \leq 2\pi.$$

Here, $g(x) = - x^{-1} \cos (\log x^{-1})$, and the total variation of $g(x)$ in the interval $[\epsilon, 2\pi]$ is given by:

$$\int_\epsilon^{2\pi} | dg/dx |\, dx = \int_\epsilon^{2\pi} | (1/x^2) [\cos(\log x^{-1}) + \sin(\log x^{-1})] |\, dx$$
$$< 2^{1/2}/\epsilon.$$

Evidently $f(x)$ is of unbounded variation in the interval $[0, 2\pi]$. It is of écart fini but is discontinuous at $x = 0$.

In the next theorem we relax somewhat the restriction on the order of the total variation of the function $g(x)$ while restricting the behavior of the function $f(x)$ in the neighborhood of the point $x = 0$.

THEOREM 7. *If $g(x)$ is a function of finite variation in the interval $[\epsilon, 2\pi]$, $0 < \epsilon < 2\pi$, its total variation being of the order $(1/\epsilon^\beta)$, and $f(x)$ is given by*

$$f(x) = f(x_0) + \int_{x_0}^x g(t)\, dt \qquad 0 < x_0 < x \leq 2\pi$$

and if $f(x)$ satisfies at $x = 0$ the condition,

$$| f(x) - f(0) | \leq A x^\alpha, \qquad \alpha > 0,$$

then, if $\beta \leq 1 + \alpha$, $f(x)$ is of écart fini.

We shall show that the integral

$$\int_a^b f(x) n \cos n(k + x)\, dx$$

is bounded; whether n is such that, first $0 < a < b \leq n^{-1/(1+\alpha)}$ or, second, $n^{-1/(1+\alpha)} \leq a < b \leq 2\pi$. In case $a < n^{-1/(1+\alpha)} < b$ the interval may be split up into two parts which come under these two cases. In the first case

$$\int_a^b f(x) n \cos n(k+x) dx = \int_a^b [f(x) - f(0)] d_x \sin n(k+x)$$
$$+ f(0)\{\sin n(k+b) - \sin n(k+a)\}.$$

The last term is bounded. The integral in the second member is not greater numerically than max $| f(x) - f(0) |$ times the total variation of $\sin n(k+x)$, when $x \leq n^{-1/(1+\alpha)}$. But max $| f(x) - f(0) | \leq Ab^\alpha \leq An^{-\alpha/(1+\alpha)}$, and the total variation of the function $\sin nx$ in an interval of length $n^{-1/(1+\alpha)}$ is of the order $n^{\alpha/(1+\alpha)}$, being at most equal to $2 + 2n^{\alpha/(1+\alpha)}/\pi$. Hence the integral in question is less than $2An^{-\alpha/(1+\alpha)} + 2A/\pi$, which is less than $3A$.

Secondly, suppose $n^{-1/(1+\alpha)} \leq a < b$. We have

$$\int_a^b f(x) n \cos n(k+x) dx = f(x) \sin n(k+x) \Big]_a^b - \int_a^b g(x) \sin n(k+x) dx.$$

The first term is evidently bounded. The second we integrate again by parts:

$$\int_a^b g(x) \sin n(k+x) dx = -\cos n(k+x) \cdot g(x)/n \Big]_a^b$$
$$+ (1/n) \int_a^b \cos n(k+x) dg(x).$$

The total variation of $g(x)$ in the interval $[n^{-1/(1+\alpha)}, 2\pi]$ is of the order $(n^{1/(1+\alpha)})^\beta$, by hypothesis, and is therefore $O(n)$ since $\beta \leqq 1+\alpha$. Consequently the last integral is bounded. A fortiori, $g(x)$ itself, taken at the limits a, b, is $O(n)$; the quantity between limits is therefore bounded. This completes the proof of the theorem.

As an example of this theorem consider the function:

$$f(x) = x^\alpha \sin x^{-\alpha}, \qquad 0 \leq x \leq 2\pi, \quad \alpha > 0.$$
$$g(x) = \alpha[x^{\alpha-1} \sin x^{-\alpha} - x^{-1} \cos x^{-\alpha}]$$
$$dg/dx = \alpha[(\alpha - 1)x^{\alpha-2} \sin x^{-\alpha} + (1-\alpha)x^{-2} \cos x^{-\alpha} - \alpha x^{-2-\alpha} \sin x^{-\alpha}]$$
$$| dg/dx | < Kx^{-2-\alpha}.$$

Consequently the total variation of $g(x)$ in the interval $[\epsilon, 2\pi]$ is less than $K(1+\alpha)(1/\epsilon^{1+\alpha})$, as required by the conditions of the theorem. Hence for every value of $\alpha > 0$, the function $f(x)$ is of écart fini.

The reader will see readily that the last two theorems could be generalized so as to cover cases in which, instead of one point in the neighborhood of which $f(x)$ is of unbounded variation, there are any finite number of such points in the interval $0 \leq x \leq 2\pi$, or even a denumerable infinitude of such points.

6. *Construction of functions of écart fini.* Having given a function of écart fini it is possible to construct another function of écart fini by multiplying the given one by a function $\alpha(x)$ of bounded variation. We have, in fact, on integrating by parts

$$\int_a^b \alpha(x)f(x)n\cos n(x+k)\,dx = \alpha(b)\int_a^b f(t)n\cos n(t+k)\,dt$$
$$-\int_a^b d\alpha(x)\int_a^x f(t)n\cos n(t+k)\,dt.$$

The first term of this expression is evidently bounded, since the quantity $\left|\int_a^b f(t)n\cos n(t+k)\,dt\right| \leq K$, a constant. The second term is not numerically greater than this constant times the total variation of $\alpha(x)$ in the interval $[a,b]$. We have thus proved

THEOREM 8. *If $f(x)$ is of écart fini, and if $\alpha(x)$ is of bounded variation, then the product $\alpha(x)f(x)$ is of écart fini.*

In a similar manner we can prove

THEOREM 9. *If $f(x)$ is a continuous function of écart fini then the function,*

$$g(x) = \int_{-\pi}^{x} \alpha(t)\,df(t), \qquad -\pi \leq x \leq \pi,$$

*is a function of écart fini, continuous in the closed interval $-\pi \leq x \leq \pi$.**

For

$$g(x) = \alpha(t)f(t)\Big]_{-\pi}^{x} - \int_{-\pi}^{x} f(t)\,d\alpha(t).$$

The term between limits is of écart fini, by the preceding theorem. The Stieltjes integral is a function of bounded variation, which is also of écart fini. The fact that $g(x)$ is continuous appears from the fact that if $\alpha(t)$ is assumed to be monotone—this will not affect the argument—we can write, by the law of the mean,

$$g(x'') - g(x') = \alpha(x'')f(x'') - \alpha(x')f(x') - f(\xi)[\alpha(x'') - \alpha(x')]$$

where $x' \leq \xi \leq x''$, since $f(x)$ is continuous. Therefore

$$g(x'') - g(x') = \alpha(x')(f(\xi) - f(x')) + \alpha(x'')(f(x'') - f(\xi)),$$

and both of these terms approach zero with $|x'' - x'|$.

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* Of course, if the definition of $g(x)$ is extended so as to render it periodic the function, thus defined, will not in general be continuous at $x = \pi$ or at congruent points.

7. *Fourier series.* In this section we shall study the order of approximation to a continuous function $f(x)$, of écart fini, by its Fourier sum S_n. For this purpose we make use of the quantity $\omega(\delta)$, called the *modulus of continuity* of $f(x)$, which is defined as follows:

$$\omega(\delta) = \max | f(x') - f(x'') |, \qquad | x' - x'' | \leq \delta.$$

For the sake of brevity we shall say that $f(x)$, of écart fini, *admits the constant K,* if K is such that the inequalities

$$\left| \int_a^b f(x+k) n \sin nx \, dx \right| \leq K, \left| \int_a^b f(x+k) n \cos nx \, dx \right| \leq K$$

are satisfied for all values of $n \geqq 1$, whether integers or not, where $| b - a | \leq 2\pi$, and where k is arbitrary.

We denote by R_n the difference $R_n = S_n - f$.

THEOREM 10. *If $f(x)$ is of écart fini, admitting the constant K, and if $| f(x) | \leq M$, then there exist positive constants A, B, independent of $f(x)$, such that*

$$| S_n(x) | \leq M(A \log \phi(n) + B) + 2K/\phi(n)$$

where $\phi(n)$ is any non-decreasing function such that $1 \leq \phi(n) \leq n$.

Consider the expression

$$S_n(x) = (1/\pi) \int_0^x [f(x+t) + f(x-t)] \frac{\sin (n + \frac{1}{2}t)}{2 \sin \frac{1}{2}t} \, dt$$

$$= (1/\pi) \left(\int_0^{\phi(n)/n} + \int_{\phi(n)/n}^{\pi} \right) [f(x+t) + f(x-t)] \frac{\sin (n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \, dt.$$

It follows that

$$| S_n(x) | \leq (M/\pi) \int_0^{\phi(n)/n} \left| \frac{\sin (n + \frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt$$

$$+ (1/\pi) \left| \int_{\phi(n)/n}^{\pi} [f(x+t) + f(x-t)] \frac{\sin (n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \, dt. \right|$$

The first integral is not greater than

$$\int_0^{1/n} \left| \frac{\sin (n + \frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt + \int_{1/n}^{\phi(n)/n} (4/t) \, dt,$$

which is less than $(2n+1)/n + 4 \log \phi(n)$. Hence the first term is not greater than

$$(M/\pi)(3 + 4 \log \phi(n)).$$

If we apply the second law of the mean to the second integral, noting that $\csc (t/2)$ is a decreasing function, we find that the second term is equal to:

$$(1/2\pi)\csc(\phi(n)/2n)\int_{\phi(n)/n}^{t_n}[f(x+t)+f(x-t)]\sin(n+\tfrac{1}{2})t\,dt,$$
$$\phi(n)/n \leq t_n \leq \pi,$$

and since f is of écart fini this is not greater, numerically, than

$$(1/2\pi)\cdot(4n/\phi(n))\cdot 2K/(n+\tfrac{1}{2}) \leq 2K/\phi(n).$$

These two results show that

$$|S_n| \leq (M/\pi)(4\log\phi(n)+3)+2K/\phi(n),$$

and our theorem is proved. As a corollary of this theorem we have

THEOREM 11. *Under the hypothesis of Theorem* 10, *there exist positive constants A, B, independent of f, such that*

$$|R_n| \leq M(A\log\phi(n)+B)+2K/\phi(n).$$

In fact

$$\begin{aligned}|R_n| &= |S_n - f| \leq |S_n| + |f| \\ &\leq M(A\log\phi(n)+(3/\pi)+1)+2K/\phi(n).\end{aligned}$$

THEOREM 12. *If $f(x)$ possesses derivatives of the first and second orders, $f'(x)$ and $f''(x)$, and if $f''(x)$ is of écart fini, admitting the constant K'', and if $|f''(x)| \leq M''$, then there exists constants $A, B, > 0$, independent of $f(x)$, such that*

$$|R_n| \leq (M''/n^2)(A\log\phi(n)+B)+4K''/n^2\phi(n),$$

where $\phi(\mu)$ is any function satisfying the inequalities

$$1 \leq \phi(n) \leq n, \qquad 1 \leq \phi(n+1)/\phi(n) \leq (n+1)/n.$$

Let

$$S_n = \sum_0^\infty A_k, \qquad A_k = a_k\cos kx + b_k\sin kx,$$

and let the Fourier sum for $f''(x)$ be denoted by

$$\begin{aligned}\sigma_n = \sum_1^n B_k &= -\sum_1^n k^2(a_k\cos kx + b_k\sin kx) \\ &= -\sum_1^n k^2 A_k.\end{aligned}$$

We have

$$\begin{aligned}R_n = \sum_{n+1}^\infty A_k = -\sum_{n+1}^\infty \frac{B_k}{k^2} &= -\sum_{n+1}^\infty \frac{1}{k^2}(\sigma_k-\sigma_{k-1}) \\ &= \frac{\sigma_n}{(n+1)^2} - \sum_{n+1}^\infty \sigma_k\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right).\end{aligned}$$

Now, by Theorem 10,

$$|\sigma_k| \leq M''(A \log \phi(k) + B) + 2K''/\phi(k).$$

Hence

$$|R_n| \leq [M''/(n+1)^2]\,(A \log \phi(n) + B) + [2K''/(n+1)^2\phi(n)]$$
$$+ BM'' \sum_{n+1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$
$$+ 2K'' \sum_{n+1}^{\infty} (1/\phi(k)) \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$
$$+ M''A \sum_{n+1}^{\infty} \log \phi(k) \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$

The third term of these five is equal to

$$[BM''/(n+1)^2].$$

The fourth term is not greater than

$$\frac{2K''}{\phi(n+1)} \sum_{n+1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = \frac{2K''}{(n+1)^2\phi(n+1)}.$$

The last term can be written

$$M''A \left(\frac{\log \phi(n+1)}{(n+1)^2} + \sum_{n+1}^{\infty} \log \frac{\phi(k+1)}{\phi(k)} \cdot \frac{1}{(k+1)^2}\right)$$
$$\leq M''A \left(\frac{\log \phi(n+1)}{(n+1)^2} + \sum_{n+1}^{\infty} \frac{1}{k(k+1)^2}\right)$$
$$\leq M''A \left(\frac{\log \phi(n+1)}{(n+1)^2} + \frac{1}{2n^2}\right)$$

since

$$\log [\phi(k+1)/\phi(k)] \leq \log [(k+1)/k] \leq 1/k.$$

Hence, uniting these results, we find

$$|R_n| \leq (M''/n^2)[2A \log \phi(n) + 2B + (A/2)] + [4K''/n^2\phi(n)],$$

and this expression is of the form specified in the theorem.

In order to prove the next theorem we shall make use of the approximating function *

$$f_\mu(x) = (1/\mu^2) \int_0^\mu dt \int_0^\mu f(x + t + t')\,dt'.$$

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* Some of the properties and applications of this kind of approximating function are given in an earlier paper of the author [*Bulletin of the American Mathematical Society*, Vol. 29, No. 6 (June, 1923)]. The present form is a natural development of a well known idea of Riemann.

It is easy to show that, if $f(x)$ is continuous,

$$(d/dx)f_\mu(x) = (1/\mu^2) \int_0^\mu [f(x+\mu+t) - f(x+t)]\, dt$$

$$(d^2/dx^2)f_\mu(x) = (1/\mu^2)[f(x+2\mu) - 2f(x+\mu) + f(x)].$$

The following facts are especially useful. If we assume that $f(x)$ is of écart fini, admitting the constant K, and that $f(x)$ is continuous and has $\omega(\delta)$ as its modulus of continuity, then

a) $f_\mu(x)$ is of écart fini and admits the constant K.
b) $(d^2/dx^2)f_\mu(x)$ is of écart fini and admits the constant $4K/\mu^2$.
c) $|(d^2/dx^2)f_\mu(x)| \leq 2\omega(\mu)/\mu^2$.
d) $|f_\mu(x) - f(x)| \leq \omega(2\mu)$.

We are now prepared to prove

THEOREM 13. *If $f(x)$ is a function of écart fini, admitting the constant K, and is continuous, having $\omega(\delta)$ as its modulus of continuity, then there exist positive constants A, B, C, independent of $f(x)$, such that*

$$|R_n| \leq \omega(1/n)(A \log \phi(n) + B) + CK/\phi(n),$$

where $\phi(n)$ is any non-decreasing function of n satisfying the inequalities

$$n \geq \phi(n) \geq 1, \quad (n+1)/n \geq \phi(n+1)/\phi(n) \geq 1.$$

We express $f(x)$ as the sum of two functions as follows:

$$f(x) = (f(x) - f_\mu(x)) + f_\mu(x).$$

Then
$$R_n = R_n' + R_n''$$

where R_n', R_n'' are the Fourier remainders of order n corresponding to $f(x) - f_\mu(x)$ and $f_\mu(x)$ respectively. Since $f(x) - f_\mu(x)$ is of écart fini, admitting the constant $2K$ we have, by Theorem 11, using the constants A, B, of Theorem 12,

$$|R_n'| \leq \omega(2\mu)(A \log \phi(n) + B) + 4K/\phi(n).$$

Since $f_\mu''(x)$, the second derivative of $f_\mu(x)$, is of écart fini, admitting the constant $4K/\mu^2$ and $|f_\mu''(x)| \leq 2\omega(\mu)/\mu^2$, we have by Theorem 12,

$$|R_n''| \leq (2\omega(\mu)/n^2\mu^2)(A \log \phi(n) + B) + (16K/n^2\mu^2\phi(n)).$$

Hence, if we choose $\mu = 1/2n$, we have,

$$|R_n''| \leq 8\omega(1/2n)(A \log \phi(n) + B) + (64K/\phi(n)),$$

and, since $\omega(1/2n) \leq \omega(1/n)$, we have

$$|R_n| \leq |R_n'| + |R_n''| \leq 9\omega(1/n)(A \log \phi(n) + B) + (68K/\phi(n)).$$

In other words, we have found constants $A, B, C, > 0$, independent of $f(x)$, such that

$$|R_n| \leq \omega(1/n)(A \log \phi(n) + B) + (CK/\phi(n)).$$

The theorem which has just been proved, together with the method of proof, was suggested by a theorem given by de la Vallée Poussin * on the order of approximation to a continuous function by its Fourier sum, and proved by a method due to Dunham Jackson.† The formula arrived at in that case is

$$|R_n| \leq \omega(1/n)(A \log n + B).$$

It is to be noted that the formula of Theorem 13 does not give any better result than the one just cited in the case where $\omega(\delta)$ is of the form $k\delta^a$ $(0 < \alpha \leq 1)$, since in this case, to apply our formula, we should have to take $\phi(n)$ of the same order as n^a (so as to make the two terms infinitesimals of about the same order). The main purpose of our theorem is to establish a formula applicable to cases in which the quantity

$$\omega(1/n) \log n$$

does not approach zero as n becomes infinite.

Given now any continuous function of écart fini we can choose the function $\phi(n)$ in such a way as to satisfy the conditions of Theorem 13 and also to satisfy the equations

$$\lim_{n=\infty} \phi(n) = \infty,$$

$$\lim_{n=\infty} \omega(1/n) \log \phi(n) = 0,$$

whence it follows that R_n approaches zero uniformly. Therefore we may state in conclusion

THEOREM 14. *The Fourier development of a continuous (periodic) function of écart fini converges uniformly to that function.*

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* C. J. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, Gauthier Villars, 1919, p. 25.

† Dunham Jackson, *Dissertation*, Göttingen, 1911, p. 40.